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A NEW CLASS OF FUNCTIONS

A. AÇIKGOZ AND S. YÜKSEL

ABSTRACT : In this paper, we introduce a new class of functions called strongly pre – I – continuous functions in ideal topological spaces. Several characterizations and some of its properties of this type of functions are obtained. We also investigate the relationship between this class of functions and other classes of non-continuous functions.

Key words and pharases : $\beta - I -$ open, semi – I – open, pre – I – open, $\beta - I -$ open set and strongly pre – I – continuous function.
2000 Mathematics Subject Classification Codes : 54C08.

1. INTRODUCTION

The purpose of the present paper is to introduce and investigate the notion of a new class of functions, namely strongly pre – I – continuous functions, and give several characterizations and their properties. Relationships between this type of functions and other classes of functions are obtained. This new class of strongly pre – I – continuous functions, which is stronger than α – pre – I – continuous [20], is a generalization of strongly $\alpha - I$ – continuous [18].

For the understanding of this paper, we should recall nine classes of functions : strongly $\alpha - I -$ continuous [18], almost – I – irresolute [19], $\alpha - I -$ irresolute [20], $\beta - I -$ irresolute [19], α – pre – I – continuous [20], almost $\alpha - I$ – irresolute [20], $\alpha - I$ – continuous [8], pre – I – continuous [6] and $\beta - I$ – continuous [8] functions in ideal topological spaces.

2. PRELIMINARIES

Throughout this paper $Cl(A)$ and $Int\ (A)$ denote the closure and the interior of A, respectively. Let (X, τ) be a topological space and I *be* an ideal of subsets of X. An ideal, I is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I) = \{x \in X \mid U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function of A with respect to I and τ [11]. We simply write A^* instead of $A^*(I)$

since there is no chance for confusion. X^* is often a proper subset of X. The hypothesis $X=X^*$ [9] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [17]. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I, \tau) = \{U \setminus I : U \in \tau \text{ and } I \in I\}$, but in general $\beta(I, \tau)$ is not always a topology [10].

Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

We recall some known definitions.

Definition 2.1. A subset S of a topological space (X, τ) is said to be *α – open* [14] (resp. *semi – open* [12], *pre – open* [13], *β– open* [1]) if $S \subset Int(Cl(Int(S)))$ (resp. $S \subset Cl(Int(S))$, $S \subset Int(Cl(S))$, $S \subset Cl(Int(Cl(S)))$).

The family of all α – open (resp. semi – open, pre – open, β – open) sets in a space X is denoted by τ^{α} (resp. SO(X), PO(X), βO(X)). It is shown in [14] that τ^{α} is is a topology for X and $\tau \subset \tau^{\alpha} \subset SO(X)$. Moreover, it is shown in [18] that $SO(X) \cup PO(X) \subset \beta O(X)$. The complement of a semi – open (resp. pre – open) set is semi – closed [4] (resp. pre – closed [13]). The interesection of all semi – closed sets containing S is called the semi – closure [4] of S and is denoted by sCl(S); the union of all semi – open sets contained in S is called the semi – interior [4] of S and is denoted by sInt(S).

Definition 2.2. A subset A of an ideal topological space (X, τ, I) is said to be *α – I – open* [7] (resp. *semi – I – open* [7], *pre – I – open* [6], *β – I – open* [7] if $A \subset Int(Cl^*(Int(A)))$ (resp. $A \subset Cl^*(Int(A)$, $A \subset Int(Cl^*(A))$, $A \subset Cl(Int(Cl^*(A)))$). The family of all α – I – open (resp. semi – I – open, pre – I – open, β – I – open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha IO(X)$ (resp. $SIO(X)$, $PIO(X)$, $\beta IO(X)$).

Definition 2.3. A function $f : (X, \tau, I) \to (Y, \upsilon)$ is called *α – I – continuous* [7] (resp. *pre – I – continuous* [6], *β – I – continuous* [7] if for every $V \in \upsilon$, $f^{-1}(V)$ is α – I – open (resp. pre – I – open, β – I – open) in (X, τ, I).

Definition 2.4. A function $f : (X, \tau, I) \to (Y, \upsilon)$ is said to be *α – I – irresolute* [20] (resp. *α – pre – I – continuous* [20], *almost α – I – irresolute* [20]) if $f^{-1}(V)$ is α – I – open (resp. pre – I – open, β – I – open) in X for every α – open set V of Y.

Definition 2.5. A function $f : (X, \tau, I) \to (Y, \upsilon)$ is said to be *β – I – irresolute* [19] (resp. *strongly α – pre – I – irresolute, strongly β – pre – I – irresolute*) if $f^{-1}(V)$ is β – I – open (resp. α – I – open, pre – I – open) in X for every β – open set V of Y.

Definition 2.6. A function $f : (X, \tau, I) \to (Y, \upsilon)$ is said to be *strongly α – I – continuous* [18] (resp. *strongly pre – I – continuous, almost I – irresolute* [19]) if $f^{-1}(V)$ is α – I – open (resp. pre – I – open, β – I – open) in X for every semi – open set V of Y.

Using the definitions above, we obtain the following diagram:

$$\begin{tikzcd}
\text{strong } \alpha - \text{pre} - I - \text{irresoluteness} \arrow[r] \arrow[d] & \text{strong } \alpha - I - \text{continuity} \arrow[r] \arrow[d] & \alpha - I - \text{irresluteness} \arrow[r] \arrow[d] & \alpha - I - \text{continuity} \arrow[d] \\
\text{strong } \beta - \text{pre} - I - \text{irresoluteness} \arrow[r] \arrow[d] & \text{strongly pre} - I - \text{continuity} \arrow[r] \arrow[d] & \alpha - \text{pre} - I - \text{continuity} \arrow[r] \arrow[d] & \text{pre} - I - \text{continuity} \arrow[d] \\
\beta - I - \text{irresoluteness} \arrow[r] & \text{almost} - I - \text{irresoluteness} \arrow[r] & \text{almost } \alpha - I - \text{irresolute} \arrow[r] & \beta - I - \text{continuity}
\end{tikzcd}$$

The following remark enables us to realize that none of the above implications is reversible.

Remark 2.7. We have the following relationships:

(a) $\alpha - I -$ irresoluteness does not imply almost $I -$ irresoluteness ([19, Example 1.2]),

(b) $\alpha - I -$ continuity does not imply almost $\alpha - I -$ irresoluteness ([19, Example 1.3])

(c) Strong $\beta -$ pre $- I -$ irresoluteness does not imply $\alpha - I -$ continuity ([20, Example 1.1]),

(d) $\beta - I -$ irresoluteness does not imply pre $- I -$ continuity ([20, Example 1.2]).

3. STRONGLY PRE - *I* - CONTINUOUS FUNCTIONS

Theorem 3.1. For a function $f : (X, \tau, I) \to (Y, \upsilon)$, the following are equivalent:

(i) f is strongly pre $- I -$ continuous;

(ii) For each $x \in X$ and each semi – open set V of Y containing f(x), there exists a pre $- I -$ open set U of X containing x such that $f(U) \subset V$;

(iii) $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}^*(f^{-1}(V)))$ for every semi – open set V of Y;

(iv) $f^{-1}(F)$ is pre $- I -$ closed in X for every semi – closed set F of Y;

(v) $\mathrm{Cl}(\mathrm{Int}^*(f^{-1}(B))) \subset f^{-1}(s\mathrm{Cl}(B))$ for every subset B of Y;

(vi) $f(\mathrm{Cl}(\mathrm{int}^*(A))) \subset s\mathrm{Cl}(f(A))$ for every subset A of X.

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $x \in X$ and let V be any semi – open set of Y containing f(x). By Definition 2.6, $f^{-1}(V)$ is pre $- I -$ open in X and contains x. Set $U = f^{-1}(V)$, then by *(i)*, U is a pre $- I -$ open subset of X containing x and $f(U) \subset V$.

(ii) $\Rightarrow$ *(iii)*. Let V be any semi – open set of Y and $x \in f^{-1}(V)$. By *(ii)*, there exists a pre $- I -$ open set U of X containing x such that $f(U) \subset V$. Thus, we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}^*(U)) \subset \mathrm{Int}(\mathrm{Cl}^*(f^{-1}(V)))$ and hence $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}^*(f^{-1}(V)))$.

(iii) ⇒ *(iv)*. Let F be any semi – closed set of Y. Set V = Y – F, then V is semi – open in Y. By (iii), $f^{-1}(V) \subset Int(Cl^*(f^{-1}(V)))$ and hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is pre – I – closed in X.

(iv) ⇒ *(v)*. Let B be any subset of Y. Since sCl(B) is a semi – closed subset of Y, by (iv) $f^{-1}(sCl(B))$ is pre – I – closed and $X - f^{-1}(sCl(B))$ is pre – I – open. Thus $X - f^{-1}(sCl(B)) \subset Int(Cl^*(X - f^{-1}(sCl(B)))) = Int(X - Int^*(f^{-1}(sCl(B)))) = X - Cl(Int^*(f^{-1}(sCl(B))))$. Therefore, we obtain $Cl(Int^*(f^{-1}(B))) \subset f^{-1}(sCl(B))$.

(v) ⇒ *(vi)*. Let A be any subset of X. By *(v)*, we have $Cl(Int^*(A)) \subset Cl(Int^*(f^{-1}(f(A)))) \subset f^{-1}(sCl(f(A)))$ and hence $f(Cl(Int^*(A))) \subset sCl(f(A))$.

(vi) ⇒ *(i)*. Let V be any semi – open of Y. Then, by *(vi)*, $f(Cl(Int^*(f^{-1}(Y - V)))) \subset sCl(f(f^{-1}(Y - V))) \subset sCl(f(f^{-1}(Y - V))) \subset sCl(Y - V) = Y - sInt(V) = Y - V$. Thus, we have $Cl(Int^*(f^{-1}(Y - V))) \subset f^{-1}(Y - V)$ and hence $X - Int(Cl^*(f^{-1}(V))) = Cl(Int^*(f^{-1}(Y - V))) \subset X - f^{-1}(V)$. Consequently, we obtain $f^{-1}(V) \subset Int(Cl^*(f^{-1}(V)))$. This shows that $f^{-1}(V)$ is pre – I – open. Therefore, f is strongly pre – I – continuous.

Lemma 3.2. ([15], [7]). Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces and $U_{\lambda i}$ be a nonempty subset of $X_{\lambda i}$ for each i = 1, 2, ..., n. Then $U = \prod_{\lambda \neq \lambda i} X_\lambda \times \prod_{i = n} U_{\lambda i}$ is a nonempty semi – open [15] (resp. pre – open [7]) subset of $\prod X_\lambda$ if and only if $U_{\lambda i}$ is semi – open (resp. pre – open) in $X_{\lambda i}$ for each i = 1, 2, ..., n.

Theorem 3.3. A function $f : (X, \tau, I) \to (Y, \upsilon)$ is strongly pre – I – continuous if the graph function $g : (X, \tau, I) \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is strongly pre – I – continuous.

Proof. Let $x \in X$ and V be any semi – open set of Y containing f(x). Then, X x V is a semi – open set of X x Y by Lemma 3.2 and contains g(x). Since g is strongly pre – I – continuous, there exists a pre – I – open set U of X containing x such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus f is strongly pre – I – continuous.

Theorem 3.4. If a function $f : (X, \tau, I) \to \prod Y_\lambda$ is strongly pre – I – continuous, then $P_\lambda of : (X, \tau, I) \to Y_\lambda$ is strongly pre – I – continuous for each $\lambda \in \Lambda$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.

Proof. Let V_λ be any semi – open set of Y_λ. Since P_λ is continuous and open, it is irresolute by Theorem 1.2 of [5] and hence $P_\lambda^{-1}(V_\lambda)$ is semi – open in $\prod Y_\lambda$. Since f is strongly pre – I – continuous, we obtain $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda of)^{-1}(V_\lambda)$ and is pre – open in X. This shows that $P_\lambda of$ is strongly pre – I – continuous, for each $\lambda \in \Lambda$.

Lemma 3.5. ([2]) Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties hold:

(1) If O is open in (X, τ, I), then $O \cap Cl^*(A) \subset Cl^*(O \cap A)$.

(2) If $A \subset X_0 \subset X$, then $Cl^*\, x_0\, (A) = Cl^*(A) \cap X_0$.

Lemma 3.6. Let (X, τ, I) be an ideal topological space. If $A \in PIO(X_0)$ and $A \subset X_0 \in PIO(X)$, then $A \in PIO(X)$.

Proof. Since $A \in PIO(X_0)$, $A \subset Int_{x_0}(Cl^*_{x_0}(A)) = X_0 \cap U$ for some $U \in \tau$. Since $X_0 \in PIO(X)$, by Lemma 3.5 (2),

$$
\begin{aligned}
A &\subset X_0 \cap U \subset U \cap \mathrm{Int}(\mathrm{Cl}^*(X_0)) \\
&= \mathrm{Int}(U \cap \mathrm{Cl}^*(X_0)) \\
&\subset \mathrm{Int}(\mathrm{Cl}^*(U \cap X_0)) \\
&= \mathrm{Int}(\mathrm{Cl}^*(Int_{x_0}(Cl^*_{x_0}(A)))) \\
&= \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(Cl^*_{x_0}(A) \cap X_0))) \\
&\subset \mathrm{Int}(\mathrm{Cl}^*(Cl^*_{x_0}(A) \cap X_0)) \\
&= \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Cl}^*(A) \cap X_0)) \\
&\subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Cl}^*(A))) \\
&\subset \mathrm{Int}(\mathrm{Cl}^*(A)).
\end{aligned}
$$

This shows that $A \in PIO(X)$.

Lemma 3.7. Let (X, τ, I) be an ideal topological space. If $A \in PI(X)$ and $X_0 \in SIO(X)$, then $A \cap X_0 \in PIO(X_0)$.

Proof. $A \cap X_0 \subset \mathrm{Int}(\mathrm{Cl}^*(A)) \cap X_0 = Int_{x_0}(\mathrm{Int}(\mathrm{Cl}^*(A)) \cap X_0)$

$$
\begin{aligned}
&\subset Int_{x_0}[\mathrm{Int}(\mathrm{Cl}^*(A)) \cap \mathrm{Cl}^*(\mathrm{Int}(X_0))] \subset Int_{x_0}[\mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(A)) \cap \mathrm{Int}(X_0))] \\
&\subset Int_{x_0}[\mathrm{Cl}^*(\mathrm{Cl}^*(A) \cap \mathrm{Int}(X_0))] \\
&\subset Int_{x_0}[\mathrm{Cl}^*(\mathrm{Cl}^*(A \cap X_0))] \\
&\subset Int_{x_0}[\mathrm{Cl}^*((A \cap X_0)] \\
&\subset Int_{x_0}[\mathrm{Cl}^*(A \cap X_0)] \cap X_0 \\
&\subset Int_{x_0}[\mathrm{Cl}^*(A \cap X_0)] \cap \subset Int_{x_0}(X_0) \\
&\subset Int_{x_0}[\mathrm{Cl}^*(A \cap X_0) \cap X_0] \\
&\subset Int_{x_0}(Cl^*X_0(A^*X_0)).
\end{aligned}
$$

This shows that $A\cap X_0 \in PIO(X_0)$.

Theorem 3.8. If $f : (X, \tau, I) \to (Y, \upsilon)$ is strongly pre – I – continuous and A is a semi – I – open subset of X, then the restriction $f / A : A \to Y$ is strongly pre – I – continuous.

Proof. Let V be any semi – open set of Y. Since f is strongly pre – I – continuous, then $f^{-1}(V)$ is pre – I – open in X. Since A is semi – I – open in X, by Lemma 3.4, $(f/A)^{-1}(V) = A\cap f^{-1}(V)$ is pre – I – open in A. Hence f/A is strongly pre – I – continuous.

Theorem 3.9. Let $f : (X, \tau, I) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda\in\Lambda\}$ be a cover of X by pre – I – open sets of (X, τ, I). Then f is strongly pre – I – continuous if $f/A_\lambda : A_\lambda \to Y$ is strongly pre – I – continuous for each $\lambda\in\Lambda$.

Proof. Let V be any semi – open set of Y. Since f/A_λ is strongly pre – I – continuous, then $(f/A_\lambda)^{-1}(V)$ is pre – I – open in A_λ. Since A_λ is pre – I – open in X, by Lemma 3.6, $(f/A_\lambda)^{-1}(V)$ is pre – I – open in X for each $\lambda \in \Lambda$. Therefore, $f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_\lambda)^{-1}(V) : \lambda \in\Lambda\}$ is pre – I – open in X because the union pre – I – open sets is a pre – I – open set [6]. Hence f is strongly pre – I – continuous.

Definition 3.10. [5] A function $f : (X, \tau, I) \to (Y, \upsilon)$ is said to be *irresolute* if $f^{-1}(V)$ is semi – open in X for every semi – open set V of Y.

Theorem 3.11. Let $f : (X, \tau, I) \to (Y, \upsilon) \to Z$ be functions, then the composition g of : $(X, \tau, I) \to Z$ is strongly pre – I – continuous if f is strongly pre – I – continuous and g is irresolute.

Proof. Let W be any semi – open subset of Z. Since g is irresolute, then $g^{-1}(W)$ is semi – open in Y. Since f is strongly pre – I – continuous, then $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is pre – I – open in X and hence gof is strongly pre – I – continuous.

Definition 3.12. [3] A space (X, τ, I) is said to be *I – submaximal* if every τ^* – dense subset of X is open in X.

Definition 3.13. [3] A space (X, τ, I) is said to be *P – I – disconnected* (briefly *P.I.d.*) if the $\emptyset \neq A^* \in \tau$ for each $A \in \tau$.

Theorem 3.14. Let (X, τ, I) be an I – submaximal and P – I – disconnected space. Then, the following are equivalent for a function $f : (X, \tau, I) \to (Y, \upsilon)$:

(i) Strongly pre – I – continuity ⇔ strong α – I – continuity.

(ii) Strong α – pre – I – irresoluteness ⇔ strong β – pre – I – irresoluteness.

(iii) α – I – irresoluteness ⇔ α – pre – I – continuity.

(iv) Continuity ⇔ α – I – continuity ⇔ pre – I – continuity.

Theorem 3.15. Let (X, τ, I) be a I – submaximal and P – I – disconnected space. Then, the following are equivalent for a function f : (X, τ, I) $\rightarrow$ (Y, υ) :

(i) Strongly pre – I – continuity $\Leftrightarrow$ α – pre – I – continuity $\Leftrightarrow$ pre – I – continuity.

(ii) Strong α – I – continuity $\Leftrightarrow$ α – I – irresoluteness $\Leftrightarrow$ α – I – continuity.

Proof of Theorem 3.14 and 3.15 follows from the fact that if (X, τ, I) is I – submaximal and P – I – disconnected space, than $\tau = \alpha IO(X) = SIO(X) = PIO(X)$ [3].

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BIONOMIC EQUILIBRIUM AND OPTIMAL HARVESTING IN A TWO SPECIES PREY-PREDATOR FISHERY MODEL WHERE PREYS FIND REFUGE IN THE SPACE

D. K. BHATTACHARYA, S. MANNA AND S. BEGUM

ABSTRACT : The paper considers the combined harvesting for infinite time horizon of a prey-predator system in which both the prey and the predator obey the law of logistic growth and some preys avoid predation by hiding. First of all, it studies how the dynamic equilibriums of the exploited system behave relative to that of the unexploited system, so long as the effort of exploitation remains positive. With this knowledge it proves geometrically the existence of a Bionomic equilibrium of the system under linear type of revenue function. Next the concept of bionomic equilibrium is applied to determine the threshold values of cost per unit effort, and the maximum value of the effort. This maximum value of the effort determines the control set. With this knowledge of the control set, the total revenue is optimized by applying Pontryagin's maximum principle under two types of depreciations, one is the standard one and the other one varies from model to model. In fact, the optimal path, the optimal biomasses and the optimal effort are obtained separately. Lastly, the results are verified by special choice of parameters.

Key words : Bionomic equilibrium, Pontryagin's maximum principle.

1. INTRODUCTION

The concept of Bionomic equilibrium as given in [7] was redefined in [2] to make it free from the shortcomings. In fact, in [2] a bionomic equilibrium is defined as the point of intersection of $L'=0$ (the locus of dynamic equilibrium) and $\pi = 0$ (the profit function) and it is denoted by (x_∞, y_∞), provided both the coordinates are positive and the effort of exploitation E remains positive at that point.

In [7] [8] the existence of Bionomic equilibrium was simple as both $L' = 0$ and $\pi = 0$ were straight lines. The existence becomes difficult if one or both of $L' = 0$ and $\pi = 0$ are nonlinear. The existence of Bionomic equilibrium under general forms of locus of dynamic equilibrium and revenue function given either by a straight line or by a plane was considered in [3], [4] [5] and [6]. The present paper adds in this direction by considering a general type of prey predator model, where the preys find refuge in the space. There is another motivation in considering this paper. In fact, in harvesting problems of infinite time horizon under the standard depreciation, the transversality condition which occurs, is not mathematically justified [1]; but still it is in practice. The present paper finds an alternative way of optimization, where such mathematically absurd condition can be avoided.

2. PREY-PREDATOR MODEL

We consider the following Prey-Predator Model

$$\text{(I)}\quad \frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \alpha y(x - k_0),\quad \frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + mu(x - k_0)y$$

under the restrictions (i) $x > K_0$, (ii) $r > \alpha L$ and (iii) $s > m \alpha K_0$, where

$x = x(t)$ = size of the prey population at time t.

$y = y(t)$ = size of the predator population at time t.

K = environmental carrying capacity of the prey.

L = environmental carrying capacity of the predator.

K_0 = Number of prey that finds refuge somewhere in space.

α = Predation parameter; m = conversion factor.

r = maximum specific growth rate of the prey.

s = maximum specific growth rate of the predator.

The model is biologically justified. Under the assumption that both the prey and the predator are subjected to the same effort E and under the restrictions (i), (ii), (iii) and (iv) [$\alpha L q_2 > s q_1$], q_1, q_2 being the catch ability coefficients of x and y respectively, the exploited form of (I) is taken as

$$\text{(II)}\quad \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha y(x - k_0) - q_1 Ex,\quad \frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + m\alpha(x - k_0) - q_2 Ey, x > k_0$$

3. DYNAMIC EQUILIBIBRIUM

Theorem 1. For the model (I), the dynamic equilibrium $(\bar{x}, \bar{y})$ occurs in the positive quadrant of the XY plane and it is asymptotically stable. Further $\bar{x} > k_0$, $\bar{y} > L$. For the exploited model (II), the dynamic equilibrium $((\bar{x})_e, (\bar{y})_e)$ for $E > 0$ occurs in the positive quadrant if E < $\frac{s - m\alpha k_0}{q_2}$. Further it is also locally asymptotically stable. Lastly, for all E > 0, if $(\bar{x})_e < \bar{x}$, then $(\bar{y})_e < (\bar{y})$.

Proof. If $A = \frac{1}{K}(rs + m\alpha^2 Lk)$, $B = s(r - \alpha L) + 2m\alpha^2 k_0 L$, $C = \alpha k_0 L(s - m\alpha k_0)$, then

$$\bar{x} = \frac{1}{2A}\left[B + (B^2 + 4AC)^{\frac{1}{2}}\right] > 0 \,. \qquad \text{..... (1)}$$

$$y = \frac{L}{s}\left[s + m\alpha(\bar{x} - k_0)\right] > 0 \qquad \text{..... (2)}$$

Obviously $\bar{x} > k_0$, $\bar{y} > L$.

Now $((\bar{x})_e, (\bar{y})_e)$ of (II) satisfies the equations

$$\frac{r(\bar{x})_e\left[1 - \frac{(\bar{x})_e}{kk}\right] - \alpha(\bar{y})_e((\bar{x})_e - k_0)}{q_1(\bar{x})_e} = \frac{s(\bar{y})_e\left[1 - \frac{(\bar{y})_e}{L}\right] - m\alpha(\bar{y})_e((\bar{x})_e - k_0)}{q_2(\bar{y})_e} = E \qquad \text{..... (3)}$$

So we have,

$$(\bar{x})_e = \frac{1}{2A}\left[(B + (\alpha Lq_2 - sq_1)E) + \{(B + (\alpha Lq_2 - sq_1)E)^2 + 4A(c - \alpha Lk_0 q_2 E)\}^{\frac{1}{2}}\right] \text{.....(4)}$$

$$(\bar{y})_e = \frac{L}{s}\left[(s - q_2 E) + m\alpha((\bar{x})_e - k_0)\right] > 0, \text{ if } E < \frac{s}{q_2} \qquad \text{..... (5)}$$

Again if (iv) holds, then

$$(\bar{x})_e > 0, \text{ if } E < \frac{c}{\alpha Lk_0 q_2} = \frac{\alpha Lk_0(s - m\alpha k_0)}{\alpha Lk_0 q_2} = \frac{s - m\alpha k_0}{q_2}.$$

As $s - m\alpha k_0 < s$, so $(\bar{x})_e > 0$, also $(\bar{y})_e > 0$, if $E < \frac{s - m\alpha k_0}{q_2}$..

Again from (2) and (5), we have $(\bar{y})_e - \bar{y} = \frac{L}{s}\left[-q_2 E + m\alpha((\bar{x})_e - \bar{x})\right]$

So, if we take $(\bar{x})_e < \bar{x}$, then for $E > 0$, we have $(\bar{y})_e < \bar{y}$.

This completes the proof.

4. BIONOMIC EQUILIBRIUM

Theorem 2.

Let the exploited prey-predator model be given by (II). Let the net profit be given by

$$\pi = p_1q_1x + p_2q_2y - c_0 \qquad .....(6)$$

where p_1, p_2 are the selling prices of unit biomass of x and y respectively and c_0 is the cost per unit effort. Then for this model there exists a unique feasible bionomic equilibrium (x_∞, y_∞). Further, the thereshold values of cost per unit effort c_0 are determined from $k_0 < x_\infty < \bar{x}$ and $L < y_\infty < \bar{y}$. Lastly, corresponding to each such value of c_0, E_{max} exists and $E_{max} < \frac{s - m\alpha k_0}{q_2}$.

Proof. The zero profit line $\pi = 0$ is given by $p_1q_1x + p_2q_2y - c_0 = 0$. To determine the locus of dynamic equilibrium L' = 0, let us write $a = \alpha Lq_2 - sq_1 > 0$, $b = \alpha Lk_0q_2 > 0$, $a' = \frac{m\alpha}{q_2} > 0$, $b' = -\frac{s}{Lq_2} <$, $c' = \frac{s - m\alpha k_0}{q_2} > 0$. Then L' = 0, is found to be a hyperbola passing through the origin whose equation is given by the general form

$$A'x^2 + 2H'xy + B'y^2 + 2G'x + 2F'y + C' = 0, \qquad .....(7)$$

$$A' = A - aa' > 0,\ H' = -\frac{ab'}{2} > 0,\ G' = \frac{-B + ab' - ac'}{2} < 0,\ F = \frac{bb'}{2} < 0,\ B' = 0 = C',$$

$$\Delta = \begin{bmatrix} A' & H' & G' \\ H' & 0 & F' \\ G' & F' & 0 \end{bmatrix} \neq 0,\ \delta = A'B' - H^2 = -H^2 < 0.$$

Let us now change the origin to the point $(\alpha', \beta') = \left(-\frac{F'}{H'}, \frac{(A'F - H'^2)}{H'^2}\right)$

Then (7) is reduced to

$$A'x'^2 + 2H'x'y' + f(\alpha', \beta') = 0 \qquad ..... (8)$$

where $f(\alpha', \beta') = A'\alpha'^2 + 2H\alpha'\beta' + 2G'\alpha' + 2F'\beta'$; and (x' y') represents the point (x, y) in the new co-ordinate system. As $\alpha' > 0$, so to find out the nature of $L' = 0$, we consider the following possibilities:

1. (i) $\alpha' > 0,\ \beta' > 0,\ f(\alpha', \beta') > 0,$
   (ii) $\alpha' > 0,\ \beta' > 0,\ f(\alpha', \beta') < 0.$
2. (i) $\alpha' > 0,\ \beta' < 0,\ f(\alpha', \beta') > 0,$
   (ii) $\alpha' > 0,\ \beta' < 0,\ f(\alpha', \beta') < 0.$

It is found that, in all cases, the asymptotes of the hyperbola are given by $x' = 0$ and $y' = \left(\frac{A'}{2H'}\right)x', (A' > 0,\ H' > 0).$

In case 1(i) (fig. 1), I' is the only branch which passes through the origin, but it is not a feasible branch, because if P (x', 0) is the point where I' meets the x'-axis, then $x'^2 = \frac{-f(\alpha', \beta')}{A'} < 0$ and so x' becomes imaginary.

In case 1(ii) (fig. 2), *I'* is the only branch passing through the origin and it is also feasible. But no portion of the branch *I'* contains points of positive quadrant. Hence the question of existence of feasible bionomic equilibrium does not arise.

In case 2(i) (fig. 1), II is the only branch, which passes through the origin, and it is also feasible. Again, in this case, feasible bionomic equilibrium like $P(x_\infty, y_\infty)$ exists, as this lies in the positive quadrant and E is positive at this point. In fact, if we choose $Q = (\bar{x}, \bar{y})$ where $E = 0$, it is found that the locus of $((\bar{x})_e, (\bar{y})_e)$ follows the are QPR so long as E remains positive.

In case 2(ii) (fig. 2), I' is the only branch that passes through the origin and it is also feasible. But no portion of this branch lies in the positive quadrant. Hence in this case also, the question of existence of bionomic equilibrium does not arise.

Thus bionomic equilibrium exists only in case 2(i). In this case, (x_∞, y_∞) is determined by the intersection of (8) and (6) (equated to zero), threshold values of c_0 are determined from the relation

$$k_0 < x_\infty < \bar{x};\ L < y_\infty < \bar{y} \qquad \text{..... (9)}$$

Corresponding to each value of c_0, kept under proper limits as determined by (9), E_{max} is obtained from (4) by putting $(\bar{x})_e = x_\infty$. To avoid complications of calculations, we obtain, the threshold values of c_0 and the maximum value E_{max} of the effort E, for special choice of parameters.

This completes the proof of the theorem.

Theorem 3. Let the exploited system be given by II. Let the standard discounted revenue be given by,

$$J = \int_0^\infty e^{-\delta t}[p_1q_1x + p_2q_2y - c_0]E(t)\,dt$$

$E \in V_t = [E_0, E_{max}]$, $E_0 = E(0)$,

Then the optimal path is given by,

$$(p_1q_1x + p_2q_2y - c_0)\left[\delta^2 + \left(\frac{rx}{K} + \frac{sy}{L} + \frac{\alpha k_0 y}{x}\right)\delta + \left\{\frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right) + m\alpha^2 y(x - k_0)\right\}\right]$$

$$-\left[\left\{\delta p_1q_1 + \frac{p_1q_1sy}{L} + m\alpha y p_2q_2\right\}\left\{rx\left(1 - \frac{x}{K}\right) - \alpha y(x - k_0)\right\}\right]$$

$$+\left[\left\{\delta p_2q_2 + \left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)p_2q_2 - \alpha(x - k_0)p_1q_1\right\}\left\{sy\left(1 - \frac{y}{L}\right) + m\alpha(x - k_0)y\right\}\right] = 0 \; .. \quad (10)$$

The optimal biomasses x_δ and y_δ are determined by the point of interesection of (7) and (10). Lastly, the optimal effort E* is obtained by using the above value of x_δ in (4).

Remark: The proof follows by applying Pontryagin's principle. But the trasversality conditions that occur are the following, viz., as t → ∞.

$$e^{\delta t}\lambda_1(t) \to \frac{\left(\delta p_1q_1 + \frac{p_1q_1sy}{L} + m\alpha y p_2q_2\right)E_{max}}{\delta^2 + \left(\frac{rx}{K} + \frac{sy}{L} + \frac{\alpha k_0 y}{x}\right)\delta + \left\{\frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right) + m\alpha^2 y(x - k_0)\right\}}$$

$$e^{\delta t}\lambda_1(t) \to \frac{\left[\delta p_2q_2 + \left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)p_2q_2 - \alpha(x - k_0)p_1q_1\right]E_{max}}{\delta^2 + \left(\frac{rx}{K} + \frac{sy}{L} + \frac{\alpha k_0 Y}{x}\right)\delta\left\{\frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right) + m\alpha^2 y(x - k_0)\right\}}$$

This type of conditions are not mathematically justified [1].

So we consider the following.

Theorem 4. Let the total discounted revenue for the exploited system (II) be given by

$$J = \int_0^\infty \frac{1}{t^{\delta+2}}(p_1q_1x + p_2q_2y - c_0)\,E(t),$$

$(p_1q_1x + p_2q_2y - c_0)\mathrm{E(t)} = (0)e^{\delta t},\ \delta > 0,\ E(t) \in V_t[E_{t_0},\ \sup(E_{\max})],\ E(t_0) = E(t_0)$
.....(11)

Then the optimal path is given by,

$$\begin{aligned}
&(p_1q_1x + p_2q_2y - c_0)\left\{m\alpha^2y(x-k_0)\right\} + \left\{m\alpha^2y(x-k_0) + \frac{sy}{L}\left(\frac{rx}{k} + \frac{\alpha k_0y}{x}\right)\right\}\\
&-\Bigl[\left\{p_2q_2m\alpha y + p_1q_1sy\right\}\left\{rx\left(1-\frac{x}{k}\right) - \alpha y(x-k_0)\right\}\\
&+\left\{\left(\frac{rx}{K} + \frac{\alpha k_0y}{x}\right)p_2q_2 - p_1q_1\alpha(x-k_0)\right\}\left\{sy\left(1-\frac{y}{L}\right) + m\alpha(x-k_0)y\right\}\Bigr] = 0 \qquad .....(12)
\end{aligned}$$

The optimal biomasses x_δ are obtained by the intersection of (7) and (12). Now using the value of x_δ in (5), the optimal effort E* can be obtained.

Proof. The Hamiltonian for (II) and (11) is given by,

$$H = \frac{(p_1q_1x + p_2q_2y - c_0)E(t)}{t^{\delta+2}} + \lambda_1(t)\left[F(x,y) - q_1Ex\right] + \lambda_2(t)\left[G(x,y) - q_2Ey\right]$$

$$= \left[\frac{(p_1q_1x + p_2q_2y - c_0)E(t)}{t^{\delta+2}} - \lambda_1q_1x - \lambda_2q_2y\right] + \lambda_1(t)F(x,y) + \lambda_2(t)\,G(x,y) \qquad .....(13)$$

where λ_1(t) and λ_2(t) are the adjoint variables to be determined suitably.

Now the adjoint equations for λ_1(t) and λ_2(t) are given by

$$\frac{d\lambda_1}{dt} = -\frac{p_1q_1E}{t^{\delta+2}} + \lambda_1(t)\left[q_1E - r + \frac{2rx}{K} + \alpha y\right] - \lambda_2(t).m\alpha y$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2q_2E}{t^{\delta+2}} + \alpha.\lambda_1(t)(x-k_0) + \lambda_2(t)\left[q_2E - s + \frac{2sy}{L} - m\alpha(x-k_0)\right]$$

Now with the help of (13), the we can reduce the above equations respectively to,

$$\frac{d\lambda_1}{dt} = -\frac{p_1 q_1 E}{t^{\delta+2}} + \lambda_1(t)\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right) - \lambda_2(t).m\alpha y \qquad \text{..... (14)}$$

$$\frac{d\lambda_2}{dt} = -\frac{p_2 q_2 E}{t^{\delta+2}} + \alpha.\lambda_1(t)(x - k_0) + \lambda_2(t)\frac{sy}{L} \qquad \text{..... (15)}$$

Solving (14) and (15), we get

$$\lambda_1(t) = A_1.e^{\alpha_1 t} + B_1.e^{\alpha_2 t}$$

$$+ \frac{\left[\left(p_2 q_2 m\alpha y + \frac{p_1 q_1 sy}{L}\right)\frac{1}{t^{\delta+2}} + \frac{(\delta+2)}{t^{\delta+3}}\left\{p_1 q_1 - p_2 q_2 m\alpha y - \frac{p_1 q_1 sy}{L}\right\} + \ldots\right] E(t)}{\left\{m\alpha^2 y(x - k_0) + \frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)\right\}} \qquad \text{..... (16)}$$

$$\lambda_2(t) = A_2.e^{\alpha_1 t} + B_2.e^{\alpha_2 t}$$

$$+ \frac{\left[\left(\frac{rx}{K} + \alpha k_0 y\right)p_2 q_2 - p_1 q_1 \alpha(x - k_0)\frac{1}{t^{\delta+2}} + \frac{(\delta+2)}{t^{\delta+3}}\left\{p_1 q_1 - p_2 q_2 m\alpha y - \frac{p_1 q_1 sy}{L}\right\} + \ldots\right] E(t)}{\left\{m\alpha^2 y(x - k_0) + \frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)\right\}}$$

where A_1, A_2 and B_1, B_2 are arbitrary constants and α_1, α_2 are suitably expressed in terms of the given parameters.

Now putting $A_1 = A_2 = 0$ and $B_1 = B_2 = 0$, and taking limit under the integral sign, we have

$$t^{\delta+2}\lambda_1(t) \rightarrow \frac{\left(p_2 q_2 m\alpha y + \frac{p_1 q_1 sy}{L}\right)E_{\max}}{\left\{m\alpha^2 y(x - k_0) + \frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)\right\}} \qquad \text{.....(17)}$$

$$t^{\delta+2}\lambda_2(t) \rightarrow \frac{\left\{\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)p_2 q_2 - p_1 q_1 \alpha(x - k_0)\right\}E_{\max}}{\left\{m\alpha^2 y(x - k_0) + \frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)\right\}} \qquad \text{.....(18)}$$

Taking H to the maximum for some $E \in V_1$ (control set), we have $\frac{\partial H}{\partial E} = 0$

So, $p_1q_1x - p_2q_2y - c_0 - t^{\delta+2}(\lambda_1q_1x + \lambda_2q_2y) = 0$.

Hence the optimal path is given by,

$(p_1q_1x - p_2q_2y - c_0)$

$$-\frac{\left[q_1x\left\{p_2q_2m\alpha y + \frac{p_1q_1sy}{L} + q_2y\right\}\left\{\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)p_2q_2 - p_1q_1\alpha(x - k_0)\right\}\right]E_{\max}}{\left\{m\alpha^2 y(x - k_0) + \frac{sy}{L}\left(\frac{rx}{K} + \frac{\alpha k_0 y}{x}\right)\right\}} = 0 \quad .....(19)$$

Consider the bionomic equilibrium (x_∞, y_∞) and using the value of E given by $E = \frac{rx\left(1 - \frac{x}{K}\right) - \alpha(x - k_0)y}{q_1x} = \frac{sy\left(1 - \frac{y}{L}\right) + m\alpha(x - k_0)y}{q_2y}$, we find that (19) reduces to (12); finally putting $x_\infty = (\bar{x})_e$ in (4), E* is obtained.

E* for special choice of parameters, and verify our results by showing that $E^* \in Vt$.

5. VERIFICATION OF THE RESULTS BY SPECIAL CHOICE OF PARAMETERS.

$r = 3,\ K = 8,\ \alpha = 2,\ k_0 = 1,\quad p_1 = 3,\ q_1 = 1$

$s = 5,\ L = 1,\ m = 1,\quad p_2 = 2,\ q_2 = 3,$

Here $r > \alpha L$, $s > m\ \alpha\ k_0$, $\alpha L q_2 > sq_1$, $A = \frac{47}{8}$, $B = 13$, $C = 6$

$\alpha' = 6 >$, $\beta' = -\frac{63}{2} < 0$, $f(\alpha', \beta') > 0$

Here $\bar{x} = \frac{4}{47}\left(13 + \sqrt{310}\right)$ and $\bar{y} = \frac{245 + 8\sqrt{310}}{235}$

$L' = 0$ is given by $25x^2 + 8xy - 48x - 48y = 0$ and $\pi = 0$ is given by $3x + 6y - c_0 = 0$. (x_∞, y_∞) is the intersection of $L' = 0$ and $\pi = 0$.

Thus $$x_\infty = \frac{-2(c_0 - 18) + 2\sqrt{(c_0 - 18)^2 + 378c_0}}{63}$$

and $$y_\infty = \frac{23c_0 - 36 - 2\sqrt{(c_0 - 18)^2 + 378c_0}}{126}$$

Using $k_0 < x_\infty < \bar{x}$, $L < y_\infty < \bar{y}$, threshold values of c_0 are found to satisfy $13.2 < c_0 < 17.67$. We choose $c_0 = 14$, then $x_\infty = 2.4$, $y_\infty = 1.11$. It follows from (4) that corresponding $E = .93$. Also putting $x = \frac{48}{25}$ and $y = 0$ in (3), we get $E_{max} = 2.28$ (approx). Thus $E_{max} \in V_t$.

Calculation of x_δ, y_δ, E^* for result of theorem 3.

We consider $\delta = 1$.

Then the equation of the optimal path is,

$$56x^2y^2 + 282x^3y - 1318x^2y + 18x^3 + 960y^3 - 72xy^2 - 90x^2 - 362xy$$
$$-1312y^2 - 112x - 224y = 0.$$

Now solving the above equation and $25x^2 + 8xy - 48x - 48y = 0$.

we get $y_\delta = 1.5$ (app). But $x_\delta = -3.5 < 0$. Hence (x_δ, x_δ) is not a feasible point, when $\delta = 1$. But this does not mean that (x_δ, x_δ) is not feasible for any $\delta > 0$. In fact, the feasible (x_δ, x_δ) may be obtained from the same result if $\delta > 0$ could be properly chosen (which can not be apprehended at random). This indicates the main drawback in assuming the discount $e^{-\delta t}$, where $\delta > 0$ is arbitrary, apart from the occurrence of mathematically unjustified transversality conditions.

Calculation of x_δ, y_δ, $E^\ddagger$ for result of theorem 4.

The equation of the optimal path is

$$564 \times 2y + 282 \times 3 - 1408 \times 2 - 816xy + 960y2 + 304x - 288y - 1120 = 0.$$

Now solving this equation and $25 \times 2 + 8xy - 48x - 48y = 0$, we get $x_\delta = 2.5$ (approx.), $y_\delta = 1.295$ (approx.). Putting this value of x_δ in (4), we get $E^* = 0.5089$ (approx.).

Hence $E^* \in V_t$.

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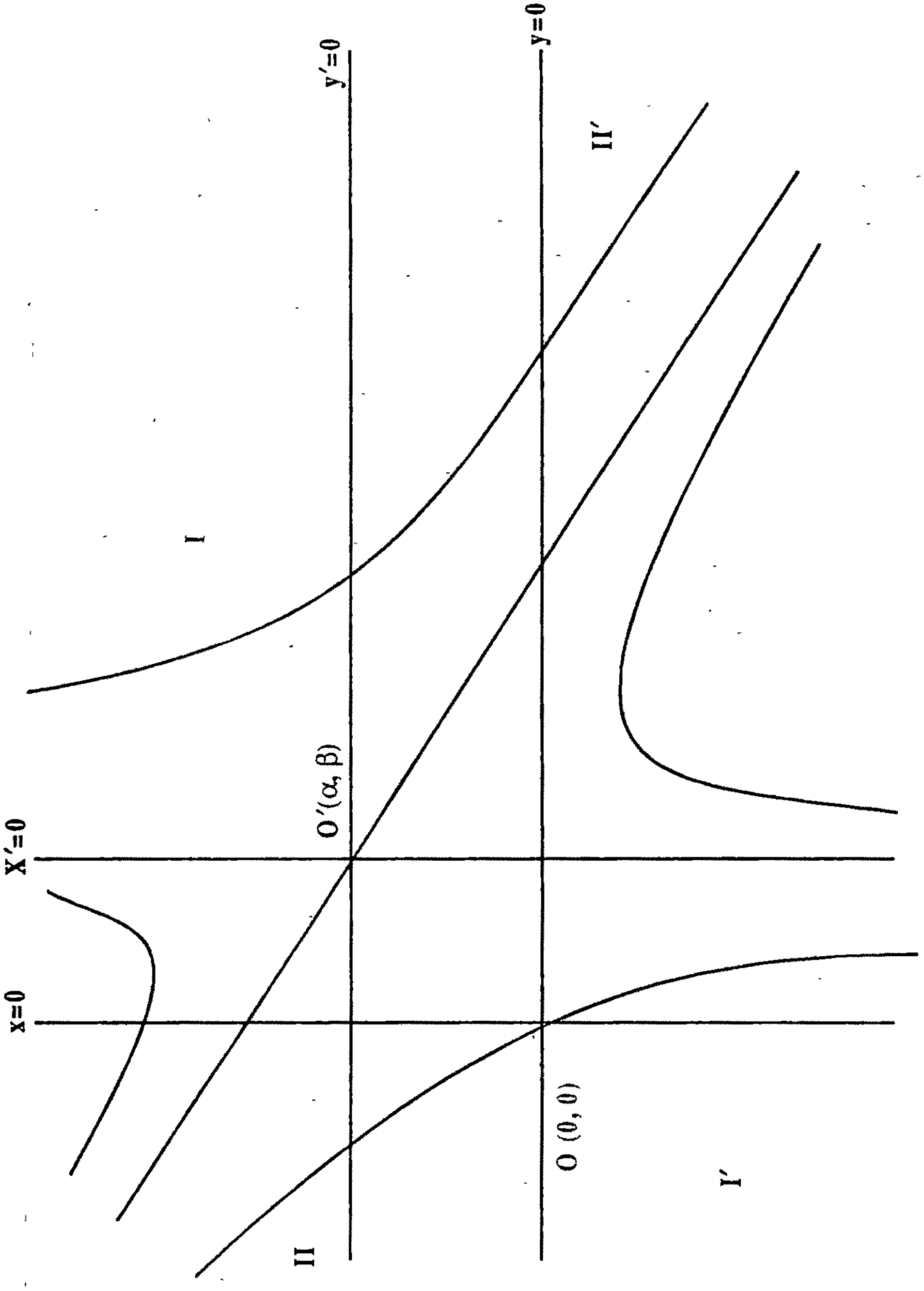

FIG. 1(i)

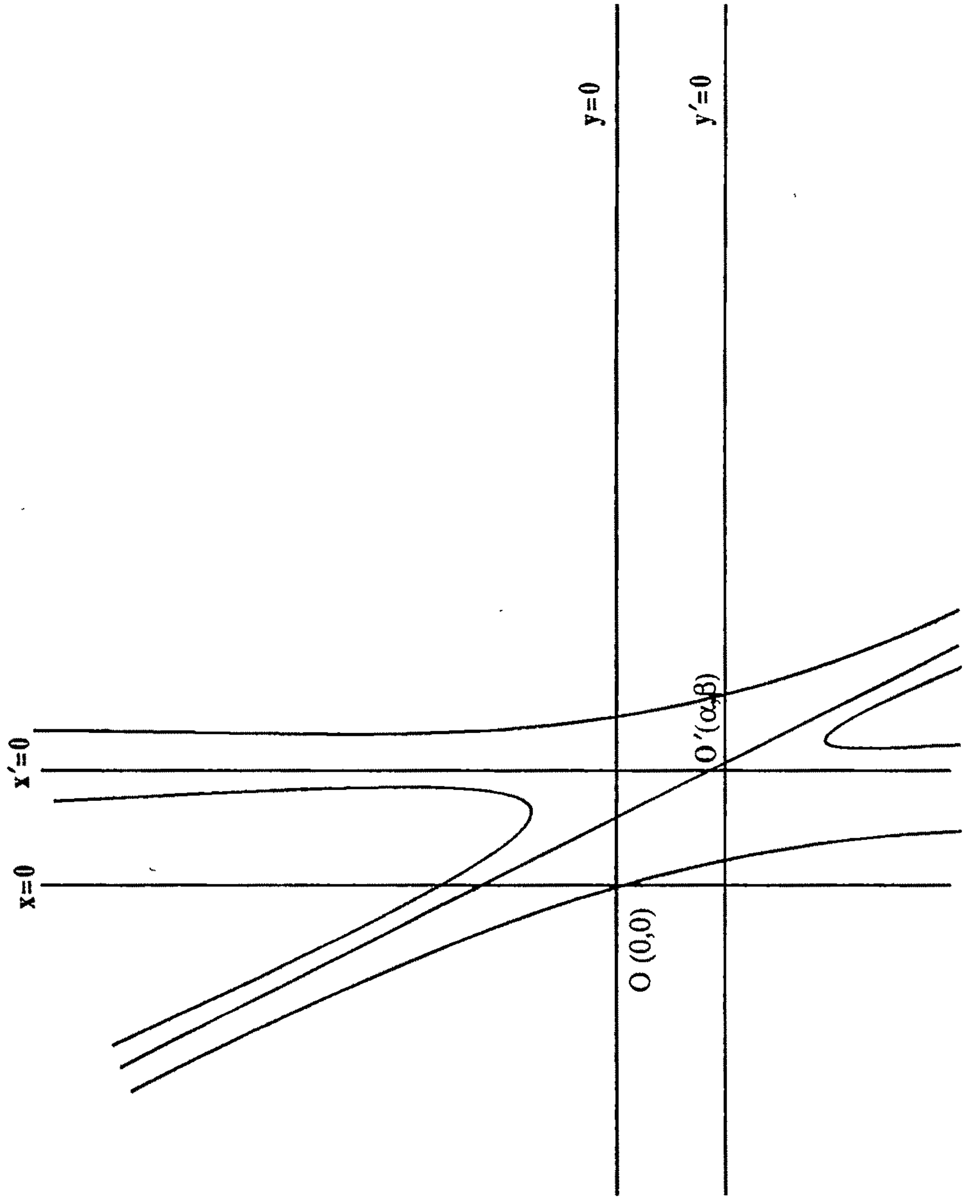

fig. 2(ii)

G-146090

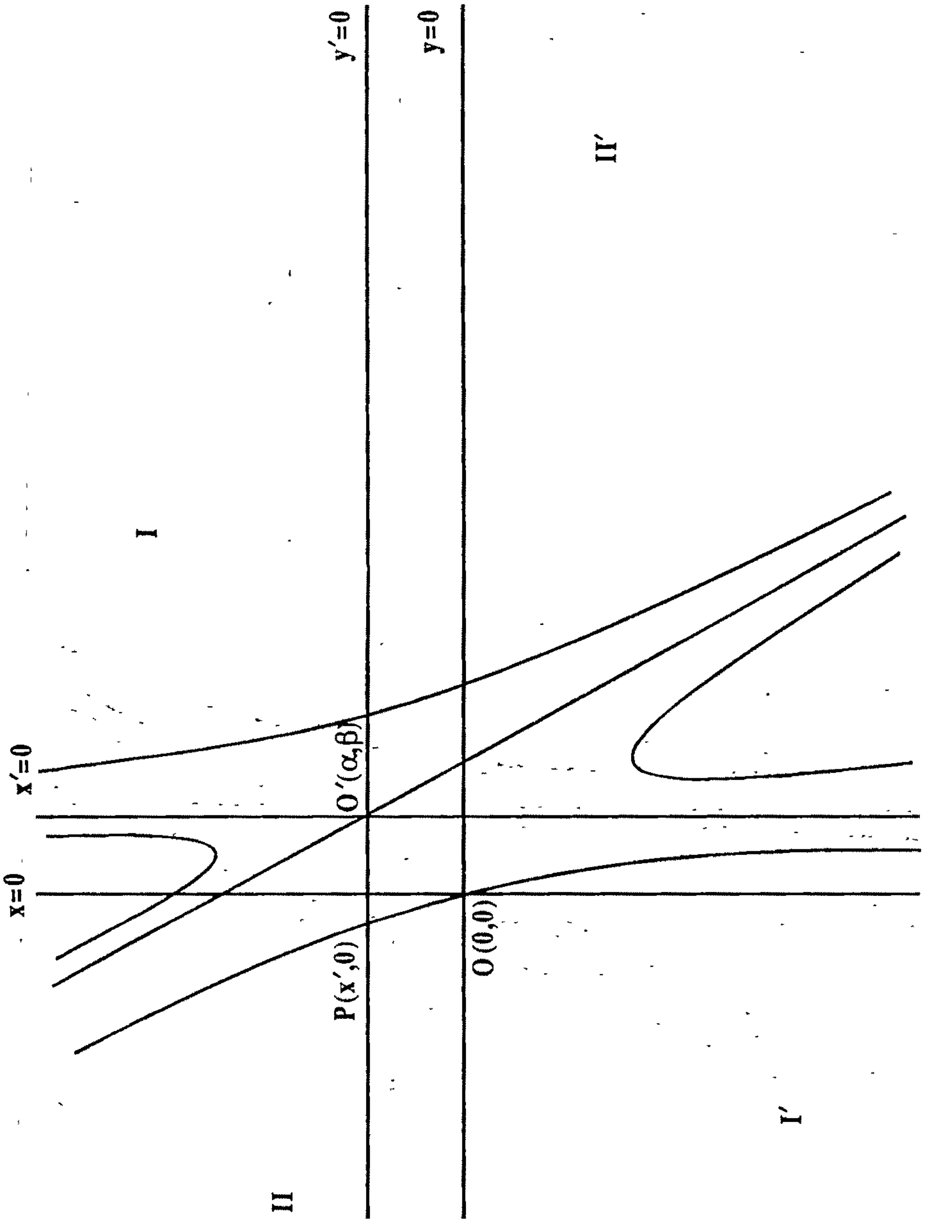

fig. 1(i)

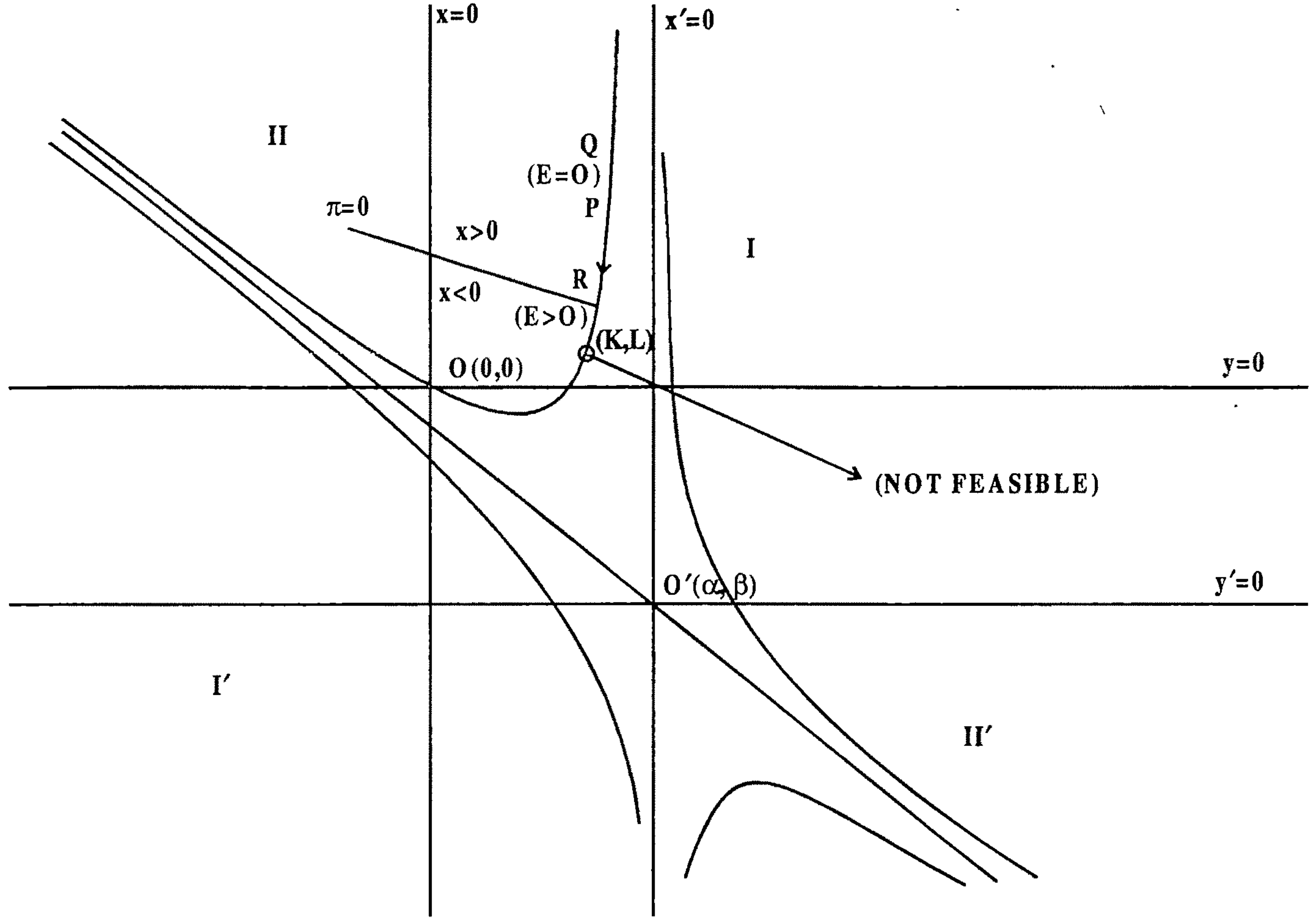

Fig. 2(i)

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ON RELATIONS BETWEEN CERTAIN TENSOR AND SEMI-SYMMETRIC SPACE

MAKOTO YAWATA AND HIDEKO HASHIGUCHI

ABSTRACT : We define the tensor $H = (H_{hijku}{}^{v})$ by

$$H_{hijku}{}^{v} = -R_{hij}{}^{a} R_{aku}{}^{v} - R_{hik}{}^{a} R_{jau}{}^{v} - R_{jkh}{}^{a} R_{aiu}{}^{v} - R_{jki}{}^{a} R_{hau}{}^{v}.$$

If the tensor H satisfies the condition $H = 0$, then Riemannian manifold (M, g) is semi-symmetric. In this paper, we also define new tensor $A = (A_{hijku}{}^{v})$ of type (1.5) by

$$A_{hijku}{}^{v} = \nabla_h \nabla_i R_{jku}{}^{v} - R_{hau}{}^{v} R_{jki}{}^{a} - \frac{1}{2} P_{hijku}{}^{v}$$

where $P = (P_{hijku}{}^{v})$ is given by

$$P_{hijku}{}^{v} = R_{hia}{}^{v} R_{jku}{}^{a} - R_{jka}{}^{v} R_{hiu}{}^{a} + \frac{1}{2}(R_{hja}{}^{v} R_{iku}{}^{a} - R_{ika}{}^{v} R_{hju}{}^{a} - R_{hka}{}^{v} R_{iju}{}^{a} + R_{ija}{}^{v} R_{hku}{}^{a}).$$

As a conclusion, we obtain some relation between A, H and semi-symmetry.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold with metric tensor g and R its Riemannian curvature tensor. A locally symmetric space satisfies the condition $\nabla_x R = 0$. From this equation, we have

(1.1) $\quad R(X, Y).R = 0$ for all tangent vectors X and Y,

where the endomorphism $R(X, Y)$ operates on R as derivation of the tensor algebra at each point of (M, g). The manifold (M, g) which satisfies the condition (1) is called a semi-symmetric space. Z.I. Szabo gives the full local classification of Riemannian semi-symmetric spaces ([5]). H.Takagi and K. Sekigawa construct counterexample ([3][6]) with respect to locally symmetric space. There are many references relating local-symmetry and semi-symmetry (for examples [1][2][4] and so on).

2. SOME NEW DEFINITIONS AND ALLIED PROPERTIES

Let us define the tensor H by

$$H(X, Y, Z, W) = (R(X, Y)R)(Z, W) + (R(Z, W)R)(X, Y),$$

and the compoments are given by

$$H_{hijku}{}^{v} = -R_{hij}{}^{a}R_{aku}{}^{v} - R_{hik}{}^{a}R_{jau}{}^{v} - R_{jkh}{}^{a}R_{aiu}{}^{v} - R_{jki}{}^{a}R_{hau}{}^{v}.$$

The condition $H(X, Y, Z, W) = 0$ is eqivalent to the condition (1.1). Also, H satisfies the following conditions

$$\begin{cases} H(X, Y, Z, W) = -H(Y, X, Z, W) \\ H(X, Y, W, Z) = -H(X, Y, Z, W) \\ H(X, Y, Z, W) + H(Y, Z, X, W) + H(Z, X, Y, W) = 0. \end{cases}$$

The tensor field H also satisfies an additional condition

$$H(X, Y, Z, W, U, V) + H(Z, W, U, V, X, Y) + H(U, V, X, Y, Z, W) = 0,$$

where we put $H(X, Y, Z, W, U, V) = g(H(X, Y, Z, W)U, V)$.

Next, we define new tensor A of type (1, 5) by

$$A(X, Y, Z, W)U = (\nabla^2 R)(Z, W, U; Y; X) - R(X, R(Z, W)Y)U - \frac{1}{2}P(X, Y, Z, W)U,$$

where $\nabla^2 R$ and P are given by

$$(2.1)\qquad \begin{cases} (\nabla^2 R)(Z, W, U; Y; X) = (\nabla_X \nabla_Y R)(Z, W)U - (\nabla_{\nabla_X Y} R)(Z, W)U, \\ P(X, Y, Z, W) = [R(X, Y), R(Z, W)] \\ + \frac{1}{2}[R(X, Z), R(Y, W)] - \frac{1}{2}[R(X, W), R(Y, Z)]. \end{cases}$$

The components of A is given by

$$A_{hijku}{}^{v} = \nabla_h \nabla_i R_{jku}{}^{v} - R_{hau}{}^{v}R_{jki}{}^{a} - \frac{1}{2}P_{hijku}{}^{v}$$

where $P = (P_{hijku}{}^{v})$ is given by

$$P_{hijku}{}^{v} = R_{hia}{}^{v}R_{jku}{}^{a} - R_{jka}{}^{v}R_{hiu}{}^{a} + \frac{1}{2}(R_{hja}{}^{v}R_{iku}{}^{a} - R_{ika}{}^{v}R_{hju}{}^{a} - R_{hka}{}^{v}R_{iju}{}^{a} + R_{ija}{}^{v}R_{hku}{}^{a}).$$

Then A has certain meaning of the decomposition of H, namely

Theorem 1. *Let (M, g) be a Riemannian manifold. Then we have*

$$A(X, Y, Z, W)U - A(Y, X, Z, W)U = H(X, Y, Z, W)U$$

for any vector fields X, Y and Z.

The proof of Theorem 1 will be given in Section·4.

Corollary 1. *If* $A(X, Y, Z, W)U = A(Y, X, Z, W)U$ for any vector fields X, Y, Z, W, U, *then* $H(X, Y, Z, W)U = 0$, *that is the Riemannian manifold* (M, g) *is a semi-symmetric space.*

Next we shall observe the properties of A;

Theorem 2. *Let* (M, g) *be a Riemannian manifold. Then the tensor* A *defined by (6) has the following properties;*

$$\begin{cases} A(X, Y, Z, W) = -A(X, Y, W, Z), \\ A(X, Y, Z, W) + A(X, Z, W, Y) + A(X, W, Y, Z) = 0, \\ A(X, Y, Z, W) + A(Z, Y, W, X) + A(W, Y, X, Z) = 0. \end{cases}$$

We also observe the similar equation for the tensor P in Section 3. We will prove Theorem 2 in Section 4.

At last we mention the relation between new tensor P and locally flatness, namely;

Theorem 3. *Let* (M, g) *be a Riemannian manifold. If the tensor* T *defined by*

$T(X, Y, Z, W)U = R(X, R(Z, W)Y)U + \frac{1}{2}P(X, Y, Z, W)U$ *satisfies the condition*

$$T_{ahjki}{}^{a} - T_{aijkh}{}^{a} + T_{ajhik}{}^{a} - T_{akhij}{}^{a} = 0,$$

then (M, g) *is locally flat.*

3. THE PROPERTIES OF THE TENSOR P

This section does not have direct relation to the proofs of theorems, but we shall observe the similar properties of the tensor P to that of the tensor A, which will be proved in Section 4.

Let P be the tensor which was already defined in (2.1). Then the symmetry and skew-symmetry of P are the following.

Proposition 1. The tensor P satisfies the following conditions;

$$\begin{cases} P(X, Y, Z, W) = -P(X, Y, W, Z), \\ P(X, Y, Z, W) + P(X, Z, W, Y) + P(X, W, Y, Z) = 0, \\ P(X, Y, Z, W) + P(Z, Y, W, X) + P(W, Y, X, Z) \\ = 2[R(X, Y), R(Z, W)] + 2[R(Z, Y), R(W, X)] + 2[R(W, Y), R(X, Z)]. \end{cases}$$

Proof. The first equation is trivial by the skew-symmetry of R. Other properties are proved by straightforward calculations as follows:

$$
\begin{aligned}
&P(X, Y, Z, W) + P(X, Z, W, Y) + P(X, W, Y, Z)\\
&= [R(X, Y), R(Z, W)] + \frac{1}{2}[R(X, Z), R(Y, W)] - \frac{1}{2}[R(X, W), R(Y, Z)]\\
&+ [R(X, Z), R(W, Y)] + \frac{1}{2}[R(X, W), R(Z, Y)] - \frac{1}{2}[R(X, Y), R(Z, W)]\\
&+ [R(X, W), R(Y, Z)] + \frac{1}{2}[R(X, Y), R(W, Z)] - \frac{1}{2}[R(X, Z), R(W, Y)] = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&P(X, Y, Z, W) + P(Z, Y, W, X) + P(W, Y, X, Z)\\
&= [R(X, Y), R(Z, W] + \frac{1}{2}[R(X, Z), R(Y, W)] - \frac{1}{2}[R(X, W), R(Y, Z)]\\
&+ [R(Z, Y), R(W, X)] + \frac{1}{2}[R(Z, W), R(Y, X)] - \frac{1}{2}[R(Z, X), R(Y, W)]\\
&+ [R(W, Y), R(X, Z)] + \frac{1}{2}[R(W, X), R(Y, Z)] - \frac{1}{2}[R(W, Z), R(X, Y)]\\
&= 2[R(X, Y), R(Z, W)] + 2[R(Z, Y), R(W, X)] + 2[R(W, Y), R(X, Z)].
\end{aligned}
$$

4. PROOFS OF THEOREMS 1 AND 2

First of all, we prove Theorem 1. By the definition of A, we have

$$
\begin{aligned}
A(X, Y, Z, W)U - A(Y, X, Z, W)U &= (\nabla^2 R)(Z, W, U; Y; X) - (\nabla^2 R)(Z, W, U; X; Y)\\
&- R(X, R(Z, W)Y)U + R(Y, R(Z, W)X)\\
&- \frac{1}{2}P(X, Y, Z, W)U + \frac{1}{2}P(Y, X, Z, W)U\\
&= (R(X, Y)R)(Z, W)U + [R(Z, W), R(X, Y)]U\\
&- R(R(Z, W)X, Y)U - R(X, R(Z, W)Y)U\\
&= (R(X, Y)R)(Z, W)U + (R(Z, W)R)(X, Y)U\\
&= H(X, Y, Z, W),
\end{aligned}
$$

which concludes Theorem 1.

Next we prove Theorem 2. By the definition, we have imediately

$$A(X, Y, Z, W) = -A(X, Y, W, Z).$$

The second equation can be seen by using the second Bianchi's identity and the symmetry of R as follows;

$$\begin{aligned}
&A(X, Y, Z, W)U + A(X, Z, W, Y)U + A(X, W, Y, Z)U \\
&\quad = (\nabla^2 R)(Z, W, U; Y; X) + (\nabla^2 R)(W, Y, U; Z; X) + (\nabla^2 R)(Y, Z, U; W; X) \\
&\qquad - R(X, R(Z, W)Y)U - R(X, R(W, Y)Z)U - R(X, R(Y, Z)W)U \\
&\qquad - \frac{1}{2}[R(X, Y), R(Z, W)]U - \frac{1}{4}[R(X, Z), R(Y, W)]U - \frac{1}{4}[R(X, W), R(Z, Y)]U \\
&\qquad - \frac{1}{2}[R(X, Z), R(W, Y)]U - \frac{1}{4}[R(X, W), R(Z, Y)]U - \frac{1}{4}[R(X, Y), R(W, Z)]U \\
&\qquad - \frac{1}{2}[R(X, W), R(Y, Z)]U - \frac{1}{4}[R(X, Y), R(W, Z)]U - \frac{1}{4}[R(X, Z), R(Y, W)]U \\
&\quad = 0.
\end{aligned}$$

The third equation is proved by direct calculation as

$$\begin{aligned}
&A(X, Y, Z, W)U + A(Z, Y, W, X)U + A(W, Y, X, Z)U \\
&\quad = (\nabla^2 R)(Z, W, U; Y; X) - R(X, R(Z, W)Y)U - \frac{1}{2}P(X, Y, Z, W)U \\
&\qquad + (\nabla^2 R)(W, X, U; Y; Z) - R(Z, R(W, X)Y)U - \frac{1}{2}P(Z, Y, W, X)U \\
&\qquad + (\nabla^2 R)(X, Z, U; Y; W) - R(W, R(X, Z)Y)U - \frac{1}{2}P(W, Y, X, Z)U \\
&\quad = (R(Z, X)R)(W, Y)U + (R(W, Z)R)(X, Y)U + (R(X, W)R)(Z, Y)U \\
&\qquad - R(X, R(Z, W)Y)U - R(Z, R(W, X)Y)U - R(W, R(X, Z)Y)U \\
&\qquad - [R(X, Y), R(Z, W)]U - [R(Z, Y), R(W, X)]U - [R(W, Y), R(X, Z)]U \\
&\quad = [R(Z, X), R(W, Y)]U - R(R(Z, X)W, Y)U - R(W, R(Z, X)Y)U \\
&\qquad + [R(W, Z), R(X, Y)]U - R(R(W, Z)X, Y)U - R(X, R(W, Z)Y)U \\
&\qquad + [R(X, W), R(Z, Y)]U - R(R(X, W)Z, Y)U - R(Z, R(X, W)Y)U \\
&\qquad - R(X, R(Z, W)Y)U - R(Z, R(W, X)Y)U - R(W, R(X, Z)Y)U \\
&\qquad - [R(X, Y), R(Z, W)]U - [R(Z, Y), R(W, X)]U - [R(W, Y), R(X, Z)]U = 0,
\end{aligned}$$

which concludes Theorem 2.

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5. PROOF OF THEOREM 3

In this section, we prove Theorem 3.

Let $T_{hijku}{}^{v}$ be a tensor field defined by

$$T_{hijku}{}^{v} = R_{hau}{}^{v}R_{jki}{}^{a} + \frac{1}{2}P_{hijku}{}^{v}$$

where $P_{hijku}{}^{v}$ is defined in (2.1). Then by the definition of $P_{hijku}{}^{v}$, we have

$$T_{ahjki}{}^{a} = R_{i}{}^{a}R_{jkha} + \frac{1}{2}(R_{h}{}^{a}R_{jkia} - R_{jka}{}^{b}R_{bhi}{}^{a}) \tag{3.1}$$

$$+ \frac{1}{4}(R_{j}{}^{a}R_{hkia} - R_{k}{}^{a}R_{hji}{}^{a} - R_{hka}{}^{b}R_{bji}{}^{a} + R_{hja}{}^{b}R_{bkl}{}^{a}).$$

If we put B_{hijk}, D_{hijk}, E_{hijk} by

$$\begin{cases} B_{hijk} = R_{habi}R_{jk}{}^{ab} - R_{habj}R_{i}{}^{ab}{}_{k} + R_{iabj}R_{h}{}^{ab}{}_{k} \\ D_{hijk} \doteq R_{habi}R_{jk}{}^{ab} - R_{habk}R_{j}{}^{ab}{}_{i} + R_{habj}R_{k}{}^{ab}{}_{i} \\ E_{hijk} = R_{h}{}^{a}R_{aijk} + R_{i}{}^{a}R_{hajk} + R_{j}{}^{a}R_{hiak} + R_{k}{}^{a}R_{hija}, \end{cases}$$

the formula (3.1) implies

$$T_{ahjki}{}^{a} - T_{aijkh}{}^{a} + T_{ajhik}{}^{a} - T_{akhij}{}^{a} = B_{hijk} + D_{hijk} + \frac{1}{4}E_{hijk}.$$

So, if we suppose $T_{ahjki}{}^{a} - T_{aijkh}{}^{a} + T_{ajhik}{}^{a} - T_{akhij}{}^{a} = 0$ which is the assumption of Theorem 3, we have

$$B_{hijk} + D_{hijk} + \frac{1}{4}E_{hijk} = 0$$

Therefore if we put

$$H_{hijk} = 4B_{hijk} - E_{hijk},$$

then

$$H_{hijk} + 4D_{hijk} + 2E_{hijk} = 0. \tag{3.2}$$

By contracting (3.2), we have

$$\frac{3}{2}R_{abcd}R^{abcd} + R_{ab}R^{ab} = 0.$$

From this formula, we obtain $R_{hijk} = 0$, namely M is locally flat.

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FIXED POINT THEOREM FOR HILBERT SPACE

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1. INTRODUCTION

The study of properties and applications of fixed points of various types of contractive mapping in Hilbert and Banach spaces were obtained among others by Browder [1], Browder and Petryshyn [2, 3] Hicks and Huffman [4], Huffman [5], Koparde and Waghmode [6]. In this paper we present a common fixed point theorem involving self mapping.

2. THEOREM

Let S, T be non-commuting self maps of a closed subset C of a Hilbert space H satisfying

$$\|STx - TSy\|^2 \le \frac{a\|y - TSy\|^2\left[1 + \|x - STx\|^2\right]}{1 + \|x - y\|^2} + b\left[\|x - STx\|^2 + \|y - TSy\|^2\right]$$

for all $x, y \in C$, a, b being positive real numbers such that $0 < a + b < 1$. Then ST, TS have a common fixed point

Proof : We define a sequence $\{x_n\}$ as follows

$ST(x_0) = x_1$, $TS(x_1) = x_2$, $ST(x_2) = x_3$, ...

In general

$ST(x_{2n}) = x_{2n+1}$, $TS(x_{2n+1}) = x_{2n+2}$ for $n = 0, 1, 2, ...$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n then we see that x_{2n} is fixed point of ST and TS therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now consider

$$\|x_{2n+1} - x_{2n+2}\|^2 = \|STx_{2n} - TSx_{2n+1}\|^2$$

$$\le \frac{a\|x_{2n+1} - TSx_{2n+1}\|^2\left[1 + \|x_{2n} - STx_{2n}\|^2\right]}{\left[1 + \|x_{2n} - x_{2n+1}\|^2\right]}$$

$$+b\left[\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2\right]$$

$$\le \frac{a\|x_{2n+1} - x_{2n+2}\|^2\left[1 + \|x_{2n} - x_{2n+1}\|^2\right]}{\left[1 + \|x_{2n} - x_{2n+1}\|^2\right]}$$

$$+b\left[\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2\right]$$

$$\le a\|x_{2n+1} - x_{2n+2}\|^2 + b\left[\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n+1} - x_{2n+2}\|^2\right]$$

$$\le (a+b)\|x_{2n+1} - x_{2n+2}\|^2 + b\|x_{2n} - x_{2n+1}\|^2$$

$$\left[1 + (a+b)\right]\|x_{2n+1} - x_{2n+2}\|^2 \le b\|x_{2n} - x_{2n+1}\|^2$$

$$\|x_{2n+1} - x_{2n+2}\|^2 \le \left(\frac{b}{a+b}\right)\|x_{2n} - x_{2n+1}\|^2$$

So in general

$$\|x_n - x_{n+1}\|^2 \le \left(\frac{b}{a+b}\right)\|x_{n-1} - x_n\|^2$$

$$\le \left(\frac{b}{a+b}\right)^n \|x_0 - x_1\|^2$$

Let $\left(\frac{b}{a+b}\right)^{\frac{1}{2}} = k$

$$\|x_n - x_{n+1}\| \le k^n\|x_0 - x_1\|$$

Now, we shall prove that $\{x_n\}$ is Cauchy sequence, for this every positive integer p we have

$$\begin{aligned}\|x_n - x_{n+p}\| &= \|x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \ldots + x_{n+p-1} - x_{n+p}\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots + \|x_{n+p-1} - x_{n+p}\| \\ &\le \left\{k^n + k^{n+1} + \ldots + k^{n+p-1}\right\}\|x_0 - x_1\| \\ &\le k^n\left[1 + k + \ldots + k^{p-1}\right]\|x_0 - x_1\| \\ &\le \frac{k^n}{(1-k)}\|x_0 - x_1\|\end{aligned}$$

as $n \to \infty$.

$\|x_n - x_{n+p}\| \to 0$ Therefore $\{x_n\}$ is a Cauchy sequence in C. As C is a closed subset of H, so there exists a point z in C such that $\lim\limits_{n\to\infty} x_n = z$.

Existence of Fixed Point

Consider

$$\begin{aligned}\|z - TSz\|^2 &= \|z - x_{2n+1} + x_{2n+1} - TSz\|^2 \\ &= \|z - x_{2n+1}\|^2 + \|x_{2n+1} - TSz\|^2 + \mathrm{Re}(z - x_{2n-1}, x_{2n+1} - TSz) \\ &= \|z - x_{2n+1}\|^2 + \frac{a\|z - TSz\|^2\left[1 + \|x_{2n+1} - STx_{2n}\|^2\right]}{1 + \|x_{2n+1} - z\|^2} \\ &\quad + b\left[\|x_{2n+1} - STx_{2n+1}\|^2 + \|z - TSz\|^2\right] + \mathrm{Re}(z - x_{2n+1}, x_{2n+1} - TSz) \\ &\le \|z - x_{2n+1}\|^2 + \frac{a\|z - TSz\|^2\left[1 + \|x_{2n+1} - x_{2n+2}\|^2\right]}{1 + \|x_{2n+1} - z\|^2} \\ &\quad + b\left[\|x_{2n+1} - x_{2n+2}\|^2 + \|z - TSz\|^2\right] + \mathrm{Re}(z - x_{2n+1}, x_{2n+1} - TSz)\end{aligned}$$

As $\{x_{2n+1}\}$, $\{x_{2n-2}\}$ are subsequences of $\{x_{2n}\}$ as $n \to \infty$ $x_{2n+1} \to z$, $x_{2n+2} \to z$ and $(z - x_{2n+1}, x_{2n+1} - TSz) \to 0$.

Therefore

$$\|z - TSz\|^2 \leq (a + b)\,\|z - TSz\|^2$$

$$1 - (a - b)\,\|z - TSz\|^2 \leq 0$$

$$\Rightarrow TSz = z$$

Similarly we can prove that $STz = z$.

This shows z is the common fixed point of ST and TS.

Uniquencess

Let us be another fixed point of ST and TS then

$$\|z - u\|^2 = \|STz - TSu\|^2$$

$$\leq \frac{a\|u - TSu\|^2\left[1 + \|z - STz\|^2\right]}{1 + \|z - u\|^2} + b\left[\|z - STz\|^2 + \|u - TSu\|^2\right]$$

$$\leq \frac{a\|u - u\|^2\left[1 + \|z - z\|^2\right]}{1 + \|z - u\|^2} + b\left[\|z - z\|^2 + \|u - u\|^2\right]$$

$$\leq 0$$

$$\Rightarrow z = u$$

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SOME COMMON FIXED POINT THEOREMS IN 2 – METRIC SPACE

SHALU SHRIVASTAVA AND SMITA NAIR

ABSTRACT : The aim of this paper is to prove common fixed point theorems in 2 – metric space.

Key words : Fixed points, 2 – metric space, asymptotic regularity of sequence, compatible maps, Intimate mappings.

1. INTRODUCTION

Gahler [1] introduced the concept of 2 – metric space. An algebric structure (X, d) is called a 2 – metric space, where X is a non empty set and d is a real valued function defined on $X \times X \times X$ satisfying the following condition.

$[M_1]$ For distinct points $x, y \in X$ there exists a point $z \in X$ Such that $d(x, y, z) \neq 0$.

$[M_2]$ $d(x, y, z) = 0$ if at least two of x, y, z are equal.

$[M_3]$ $d(x, y, z) = d(x, z, y) = d(y, z, x)$

$[M_4]$ $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z) \; \forall \; x, y, z, a \in X$

We first state some useful definitions.

Definition 1: A sequence $\{x_n\}$ in a 2 – metric space (X, d) is said to be convergent to a point $x \in X$ denoted by

$$\lim_{n \to \infty} x_n = x \quad \text{if} \quad \lim_{n \to \infty} d(x_n, y, z) = 0$$

for all $z \in X$; the point X is called the limit of the sequence $\{x_n\}$ in X.

Definition 2: A sequence $\{x_n\}$ in a 2 – metric space (X, d) is called a Cauchy sequence if $m, n \lim_{n \to \infty} d(x_m, x_n, z) = 0$ for all $z \in X$.

Definition 3: A 2 – metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 4: For self mappings S and T of a 2 – metric space (X, d), a sequence $\{x_n\}$ in X is called asymptotic regular with respect to the pair

(S, T) if $\lim_{n \to \infty} d(Sx_n, Tx_n, a) = 0$ for all $a \in X$.

Definition 5: The pair (S, T) is called compatible if $\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0$ for all $a \in X$.

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

The concept of asymptotic regularity for a pair of maps was introduced by Rhoades et al [4]. Also the concept of compatible mappings was introduced by Jungck [3]. Singh S. L. and Virendr [6] have proved a common fixed point theorem in complete 2 – metric space. Using the concept of compatible mappings, we prove the following fixed point theorem.

In 1976, Jungck [2] gave a common fixed point theorem for commuting maps which generalizes Banach's fixed point theorem. This result was further generalized and extended in various ways by many authors. Further Jungck [3] introduced more generalized concept of commutativity, so called compatibility, which is more general than that of weak, commutativity.

Recently Sahu, Dhagat and Srivastava [5] generalized the concept of compatible mappings of type (A) so called intimate mappings.

Definition 6: Let (X, d) be a 2 – metric space. A, S be two self maps defined on X. Then the pair (A, S) is said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx, a) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx, a) = 0$ $\forall\ a \in X$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 7: Let A, S be self maps of 2 – metric space (X, d). The pair (A, S) is said to be S – intimate iff $\alpha\, d(SAx_n, Sx_n, a) \le \alpha\, d(AAx_n, Ax_n, a)$ where α = lim sup or lim inf, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Preposition 2.1.: If the pair (A, S) is compatible of type (A) then it is both A and S intimate.

Proof: Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Then

$$d(ASx_n, Ax_n, a) \le d(ASx_n, SSx_n, a) + d(ASx_n, Ax_n, SSx_n) + d(SSx_n, Ax_n, a)$$

$$d(ASx_n, Ax_n, a) \le d(SSx_n, Ax_n, a) \le d(SSx_n, Ax_n, Sx_n) + d(Sx_n, Ax_n, a) + d(SSx_n, Sx_n, a)$$

$$\alpha d(ASx_n, Ax_n, a) \le \alpha.0 + \alpha.0 + \alpha.d(SSx_n, Sx_n, a)$$

Implies

$\alpha\, d(ASx_n, Ax_n, a) \leq \alpha\, d(SSx_n, Sx_n, a)$

Thus the pair (A, S) is A – intimate. Similarly we can prove that (A, S) is S – intimate.

Preposition 2.2.: A and S be self maps of a 2 – metric space (X, d). If the pair is S – intimate and $At = St = p \in X$ for some $t \in X$.

Then $d(Sp, p, a) \leq d(Ap, p, a)\ \forall\ a \in X$.

Proof: Suppose $x_n = t$ for all $n \geq 1$

So $Ax_n = Sx_n \rightarrow At = St = p$

Since the pair is S – intimate then

$$\begin{aligned} d(SAt, St, a) &= \lim_{n\to\infty} d(SAx_n, Sx_n, a) \\ &\leq \lim_{n\to\infty} d(AAx_n, Ax_n, a) \\ &= d(AAt, At, a) \end{aligned}$$

implies $d(Sp, p, a) \leq d(Ap, p, a)$

Main Result 1: Let P, S, T be self mappings of a complete 2 – metric space (X, d) satisfying

(3.1) $d(Px, Py, a) \leq a_1 d(Sx, Px, a) + a_2 d(Ty, Py, a)\ \forall\ x, y, a \in X$.

where a_1, a_2 are non - negative numbers such that $a_1 < 1$, $a_2 < 1$.

(3.2) The pair (P, S), (P, T) are S – intimate and T – intimate respectively.

(3.3) $P(X) \subset S(X)$, $P(X) \subset T(X)$.

Then P, S, T have a unique common fixed point.

Proof: Let $x_0 \in X$ then by (3.3) $\exists$ a pt $x_1 \in X$. s.t. $Tx_1 = Px_0$, for this pt x, $\exists$ a pt $x_2 \in X$ s.t. $Sx_2 = Px_1$ inductively we can define a sequence $\{y_n\}$ in X as

$$y_{2n} = Tx_{2n+1} = Px_{2n}$$
$$y_{2n+1} = Sx_{2n+2} = Px_{2n+1} \quad n = 0, 1, 2, \ldots$$

We now prove that $\{y_n\}$ is a Cauchy sequence.

Considering (4.1) we have

$d(y_{2n}, y_{2n+1}, a) = d(Px_{2n}, Px_{2n+1}, a)$

$$\leq a_1 d(Sx_{2n}, Px_{2n}, a) + a_2 d(Tx_{2n+1}, Px_{2n+1}, a)$$

$$\leq a_1 d(y_{2n-1}, y_{2n}, a) + a_2 d(y_{2n}, y_{2n+1}, a)$$

$$d(y_{2n}, y_{2n+1}, a) \leq \left(\frac{a_1}{1-a_2}\right) d(y_{2n-1}, y_{2n}, a)$$

$$d(y_{2n}, y_{2n+1}, a) \leq h d(y_{2n-1}, y_{2n}, a)$$

where $h = \frac{a_1}{1-a_2} < 1$.

Now,

$$d(y_n, y_{n+1}, a) \leq h d(y_{n-1}, y_n, a) \leq \ldots \leq h^n d(y_0, y_1, a)$$

For every integer p > 0

$$d(y_n, y_{n+p}, a) \leq h^n[1 + h + \ldots + h^{p-2}]\, d(y_0, y_1, a) + h^n[1 + h + \ldots + h^{p-2}]\, d(y_0, y_1, y_{n+p})$$

$$\leq h^n[1 + h + \ldots + h^{p-2}]\{d(y_0, y_1, a) + d(y_0, y_1, y_{n+p})\}$$

$$\leq \frac{h^n}{(1-h)}[d(y_0, y_1, a) + d(y_0, y_1, y_{n+p})]$$

as $n \to \infty$

$d(y_n, y_{n+p}, a) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence in X. Let $\{y_n\}$ converge to a point p in X. Then $Px_{2n}, Tx_{2n+1}, Sx_{2n+2} \to p$

Let $\exists\ u \in X$ such that $Su = p$.

From (3.1)

$$d(Pu, Px_{2n}, a) \leq a_1 d(Su, Pu, a) + a_2 d(Tx_{2n}, Px_{2n}, a)$$

as $n \to \infty$

$$d(Pu, P, a) \leq a_1 d(p, Pu, a) + a_2 d(p, p, a)$$

$$d(Pu, p, a) \leq a_1 d(Pu, p, a)$$

Which is a contradiction. Hence $Pu = p$. As $P(X) \subset T(X)\ \exists\ v \in X$

such that $Tv = p$. From (4.1) $d(Pu, Pv, a) \leq a_1 d(Su, Pu, a) + a_2 d(Tv, Pv, a)$ $d(p, Pv, a) \leq a_2 d(p, Pv, a)$ which is a contradiction. Hence $Pv = p$.

As $Su = Pu = p$. and the pair is S - intimate so by (3) we have

$$d(Sp, p, a) \leq d(Pp, p, a).$$

Suppose $Pp \neq p$ then from (4.1)

$$\begin{aligned} d(Pp, p, a) &= d(PPu, Pv, a) \\ &\leq a_1 d(SPu, PPu, a) + a_2 d(Tv, Pv, a) \\ &\leq 2a_1 d(Pp, p, a) \end{aligned}$$

Which is a contradiction. Similarly $Tp = Pp = p$.

Hence, S, P, T have same fixed point.

Uniqueness of Fixed Point.

Let us consider another fixed point z such that, $Tz = Sz = Pz = z$.

Then

$$\begin{aligned} d(p, z, a) &= d(Pp, Pz, a) \\ &\leq a_1 d(Sp, Pp, a) + a_2 d(Tz, Pz, a) \end{aligned}$$

$d(p, z, a) = 0 \ \forall \ a \in X$

This shows $p = z$.

2. THEOREM

Let A, S and T be self mappings of a complete 2 – metric space, (X, d) satisfying the following conditions.

$$d(Ax, Ay, a) \leq \frac{a_1 d(Ty, Ay, a)\left[1 + d(Sx, Ax, a)\right]}{1 + d(Sx, Tx, a)}$$

$$+ a_2 d(Sx, Ax, a) + a_3 d(Ty, Ay, a) + a_4 d(Sy, Ay, a) \qquad \text{..... (1.1)}$$

for all x, y, a in X, where a_1, a_2, a_3 and a_4 are non negative numbers such that

$$a_1 + a_2 + a_3 + a_4 < 1$$

The pair (A, S) and (A, T) are compatible ..... (1.2)

There exists a sequence $\{x_n\}$ which is asymptotically regular with respect to (A, S) and (A, T)

S and T are continuous. ..... (1.3)

If d is continuous, then A, S and T have a unique fixed point. ..... (1.4)

Proof: Let $\{x_n\}$ satisfy (1.3) from (1.1) we have

$$d(Ax_n, Ax_m, a) \le \frac{a_1 d(Tx_m, Ax_m, a)\left[1 + d(Sx_n, Ax_n, a)\right]}{1 + d(Sx_n, Tx_n, a)}$$

$$+\ a_2 d(Sx_n, Ax_n, a) + a_3 d(Tx_m, Ax_m, a) + a_4 d(Sx_m, Ax_m, a)$$

Making $m, n \to \infty$ and using (1.3) we get

$\lim_{m,n\to\infty} d(Ax_n, Ax_m, a) = 0$ for all a in X.

Hence $\{Ax_n\}$ is a Cauchy sequence and so converges to some z in X (As X is complete).

Also,

$$d(Sx_n, z, a) \le d(Sx_n, Ax_n, a) + d(Ax_n, z, a) + d(Sx_n, z, Ax_n)$$

Making $n \to \infty$ and using (1.3), we have

$\lim_{n\to\infty} d(Sx_n, z, a) = 0$, So $Sx_n \to z$

Similarly $Tx_n \to z$

Now from (1.4) we have

$SAx_n \to Sz,\ S^2x_n = SSx_n \to Sz,\ STx_n \to Sz,$

$TAx_n \to Tz,\ T^2x_n = TTx_n \to Tz,\ TSx_n \to Tz$

Also from (1.2), we have

$$d(ASx_n, Sz, a) \le d(ASx_n, SAx_n, a) + d(SAx_n, Sz, a) + d(ASx'_n, SAx_n, Sz)$$

$d(ASx_n, Sz, a) \to 0,$ *as* $n \to \infty$

So $ASx_n \to Sz$

Similarly $ATx_n \to Tz$

Now from (1.1), we get

$$d(ASx_n, ATx_n, a) \le \frac{a_1 d(TTx_n, ATx_n, a)\left[1 + d(SSx_n, ASx_n, a)\right]}{1 + d(SSx_n, TSx_n, a)}$$

$$+\ a_2 d(SSx_n, ASx_n, a) + a_3 d(TTx_n, ATx_n, a) + a_4 d(STx_n, ATx_n, a)$$

$$= \frac{a_1 d\left(T^2x_n, ATx_n, a\right)\left[1+d\left(S^2x_n, ASx_n, a\right)\right]}{1+d\left(S^2x_n, TSx_n, a\right)}$$

$$+ a_2 d(S^2x_n, ASx_n, a) + a_3 d(T^2x_n, ATx_n, a)$$

$$+ a_4 d(STx_n, ATx_n, a) \quad .... \text{(i)}$$

Further

$$\begin{aligned} d(STx_n, TSx_n, a) &\le d(STx_n, ASx_n, a) + d(STx_n, TSx_n, ASx_n) + d(ASx_n, TSx_n, Sz) \\ &\le d(STx_n, ASx_n, a) + d(STx_n, TSx_n, ASx_n) + d(ASx_n, ATx_n, a) \\ &\quad + d(ATx_n, TSx_n, a) + d(ASx_n, TSx_n, ATx_n) \end{aligned}$$

i.e.

$$d(STx_n, TSx_n, a) \le d(STx_n, ASx_n, a) + d(STx_n, TSx_n, ASx_n)$$

$$+ \frac{a_1 d\left(T^2x_n, ATx_n, a\right)\left[1+d\left(S^2x_n, ASx_n, a\right)\right]}{1+d\left(S^2x_n, TSx_n, a\right)} + a_2 d(S^2x_n, ASx_n, a)$$

$$a_3 d(T^2x_n, ATx_n, a) + a_4 d(STx_n, ATx_n, a) + d(ATx_n, TSx_n, a)$$

$$+ d(ASx_n, TSx_n, ATx_n) \quad [\textit{from (i)}]$$

Making $n \to \infty$ we get

$$d(Sz, Tz, a) \le d(Sz, Sz, a) + d(Sz, Tz, Sz) + \frac{a_1 d(Tz, Tz, a)\left[1+d(Sz, Sz, a)\right]}{1+d(Sz, Tz, a)}$$

$$+ a_2 d(Sz, Sz, a) + a_3 d(Tz, Tz, a) + a_4 d(Sz, Tz, a)$$

$$+ d(Tz, Tz, a) + d(Sz, Tz, Tz)$$

i.e. $d(Sz, Tz, a) \le a_4 d(Sz, Tz, a)$

It follows that $Tz = Sz$.

Again from (1.1) we have

$$d(ATx_n, Az, a) \le \frac{a_1 d(Tz, Az, a)\left[1+d(STx_n, ATx_n, a)\right]}{1+d(STx_n, TTx_n, a)} + a_2 d(STx_n, ATx_n, a)$$

$$+ a_3 d(Tz, Az, a) + a_4 d(Sz, Az, a)$$

Making $n \to \infty$, we get

$$d(Tz, Az, a) \leq \frac{a_1 d(Tz, Az, a)\left[1 + d(Sz, Tz, a)\right]}{1 + d(Sz, Tz, a)} + a_2 d(Sz, Tz, a) + a_3 d(Tz, Az, a) + a_4 d(Sz, Az, a)$$

$$= (a_1 + a_2 + a_3) d(Tz, Az, a)$$

i.e.

$$d(Tz, Az, a) \leq (a_1 + a_2 + a_3) d(Tz, Az, a)$$

If follows that $Az = Tz$ Thus $Sz = Tz = Az$

Also from (1.1) we have

$$d(AAz, Az, a) \leq \frac{a_1 d(Tz, Az, a)\left[1 + d(SAz, AAz, a)\right]}{1 + d(SAz, TAz, a)} + a_2 d(SAz, AAz, a) + a_3 d(Tz, Az, a) + a_4 d(Sz, Az, a)$$

$$= a_2 d(SAz, AAz, a)$$

$$= a_2 d(SAz, ASz, a)$$

$$= 0$$

Hence, $AAz = Az = u$ (Say). And

$$d(Su, u, a) = d(SAz, u, a)$$

$$\leq d(SAz, ASz, a) + d(SAz, ASz, a) + d(ASz, u, a)$$

$$= 0$$

Thus $Su = u$ similarly $Tu = u$. Thus $Au = Su = Tu = u$ i.e. u is a common fixed point of A, S and T.

To prove the uniqueness of u, let v be another common fixed point of A, S and T. Then from (1.1) we have

$$d(Au, Au, a) \leq \frac{a_1 d(Tv, Av, a)\left[1 + d(Su, Au, a)\right]}{1 + d(Su, Tu, a)} + a_2 d(Su, Au, a) + a_3 d(Tv, Av, a) + a_4 d(Sv, Av, a)$$

$$= 0$$

Hence $u = v$

This completes the proof.

3. COROLLARY

Taking $a_3 = a_2$ we can get the result proved in [7].

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S-CLUSTER SETS AND S-CLOSEDNESS IN BITOPOLOGICAL SPACES

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ABSTRACT : In the present paper, a type of functions between two bitopological spaces is introduced and studied. The concept is developed to some extent and is ultimately applied to the characterizations of bitopological *S*-closedness; in the process certain formulations of *S*-closedness and Hausdorffness of bitoplogical spaces are obtained.

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1. INTRODUCTION

The study of cluster sets for functions and multifunctions was initiated long ago, mainly with an eye to the investigations of the theory of real and complex functions, certain details of which can be found in the classical book of Collingwood and Lohwer [2]. The notion of cluster sets for functions and multifunctions between arbitrary topological spaces was first introduced by Weston [16], followed by many others, for studying compactness and some other important variant forms like near compactness, quasi *H*-closedness etc. (e.g., see [3, 4, 8]).

Compactness and many of its allied forms including those mentioned above have already been generalized to bitopological spaces [6] by many topologists. Successful attempts have also been noticed towards the extensions of the concepts of cluster sets to bitopological context along with certain applications to the characterizations of bitopological near compactness and certain separation properties (for details one may refer to [9, 10]). In this paper, we aspire to introduce a new kind of cluster sets of functions between two bitopological spaces, to ultimately characterize *S*-closedness of a bitopological space via such cluster sets.

In Section 2, we introduce bitopological *S*-cluster sets and formulate pairwise Hausdorffness [6] by means of them; certain degeneracy conditions for such cluster sets are also worked out. In Section 3, our aim of characterizing bitopological *S*-closedness in terms of the said cluster sets is accomplished after doing a few results as supporting tools.

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In what follows, by a space $X(Y)$ we shall mean the bitopological space (X, Q_1, Q_2) (resp. (Y, P_1, P_2)), where Q_1, Q_2 (resp. P_1, P_2) are two arbitrary topologies on X (resp. Y), and the notation $f : X \to Y$ will denote a function from the space X to the space Y. For a subset A of the space (X, Q_1, Q_2), Q_i-cl$A$ and Q_i-int$A$ will stand for respectively the closure and interior of A in (X, Q_i), for $i = 1, 2$. A subset A of a space X is called ij-semiopen [1] (ij-regular open [13]) if for some Q_i-open set U, $U \subset A \subset Q_j$-cl$U$ (resp. $A = Q_i$-intQ_j-cl$A$), where (and hereafter in every such statement involving both i and j) we assume $i, j \in \{1, 2\}$ and $i \neq j$. The collection of all ij-semiopen sets of X will be denoted by $SO_{ij}(X)$, while the family of all those ij-semiopen sets of X, each containing a subset A of X, will be denoted by $SO_{ij}(A)$. In particular, when $A = \{x\}$ for some $x \in X$, we write $SO_{ij}(x)$ instead of $SO_{ij}(\{x\})$. A point x of X is said to be in the ij-θ-closure of a subset A of X, written as $x \in ij$-θ-cl$A$, if for each Q_i-open neighbourhood (henceforth nbd, for short) U of x, $(Q_j$-cl$U) \cap A \neq \phi$ [5]. $A(\subseteq X)$ is called ij-θ-closed if $A = ij$-θ-cl$A$. Clearly Q_i-cl$A \subseteq ij$-θ-cl$A$, for any $A \subseteq X$, while the equality is valid i.e., Q_i-cl$A = ij$-θ-cl$A$ holds if $A \in Q_j$ [5]. For two spaces X and Y, the cartesian product set $X \times Y$ will always be considered to be endowed with the topologies T_1 and T_2, where $T_i = Q_i \times P_i$. A nonvoid collection $\mathcal{G}$ of nonempty subsets of X is called a grill [15] on X if (i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A, B \subseteq X$ with $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

2. BITOPOLOGICAL *S*-CLUSTER SET, IN GENERAL

Having furnished some prerequisites in the last section, we now consider the following definition (we note that (b) and (c) for filters also appear in [7] under different terminologies, notations and forms).

Definition 2.1. Let $\mathcal{F}$ be a filterbase or a grill on a space (X, Q_1, Q_2) and $A \subseteq X$.

(a) The *ij-θ_s-closure of* A, denoted by ij-θ_s-cl$A$, is defined to consist of precisely those points x of X such that Q_j-cl$U \cap A \neq \phi$, for each $U \in SO_{ij}(x)$. A is called *ij-θ_s-closed* if $A = ij$-θ_s-cl$A$.

(b) The *ij-θ_s-adherence* of $\mathcal{F}$, written as ij-θ_s-ad$\mathcal{F}$, is defined by ij-θ_s-ad$\mathcal{F} = \cap\{ij$-θ_s-cl$F : F \in \mathcal{F}\}$.

(c) $\mathcal{F}$ is said to *ij-θ_s-converge to a point* $x \in X$, in notation $\mathcal{F}$ ij-θ_s x if for each $U \in SO_{ij}(x)$, there exists some $F \in \mathcal{F}$ such that $F \subseteq Q_j$-cl$U$.

Definition 2.2. The *ij-S-cluster set* $S_{ij}(f, x)$ of a function $f : X \to Y$ at a point $x \in X$ is defined to be the set $\cap\{ji$-θ-cl$f(Q_j$-cl$U) : U \in SO_{ij}(x)\}$.

We characterize the above concept of ij-S-cluster sets in the next theorem.

Theorem 2.3. *Let $f : X \rightarrow Y$ be a function and $x \in X$. Then the following are equivalent:*

(a) $y \in S_{ij}(f, x)$.

(b) $\mathcal{B} = \{f^{-1}(P_i - cl\, V) : y \in V \in P_j\}$ *is a filterbase on X, which ij–θ_s–adheres at x.*

(c) *There exists a grill Ω on X such that $\Omega \xrightarrow{ij-\theta s} x$ and $y \in \cap\{ji\text{–}\theta\text{–}clf(G) : G \in \Omega\}$.*

Proof. (a) $\Rightarrow$ (b): That $\mathcal{B}$ is a filterbase on X can easily be seen. Now $y \in S_{ij}(f, x) \Rightarrow y \in \cap\{ji\text{–}\theta\text{–}\mathrm{cl}f(Q_j\text{–}\mathrm{cl}U) : U \in S_{ij}(x)\} \Rightarrow$ for each P_j-open nbd V of y and for each $U \in SO_{ij}(x)$, $P_i - \mathrm{cl}V \cap f(Q_j - \mathrm{cl}U) \neq \phi$, i.e., $f^{-1}(P_i - \mathrm{cl}V) \cap Q_j - \mathrm{cl}U \neq \phi \Rightarrow x \in ij - \theta_s - \mathrm{ad}\mathcal{B}$.

(b) $\Rightarrow$ (c): Let $\mathcal{F}$ be the filter generated by $\mathcal{B}$. Then $\Omega = \{G \subseteq X : G \cap F \neq \phi$, for each $F \in \mathcal{F}\}$ is a grill on X. By hypothesis, for each $U \in SO_{ij}(x)$ and for all P_j-open nbd V of y, we have $Q_j - \mathrm{cl}U \cap f^{-1}(P_j - \mathrm{cl}V) \neq \phi$. Thus $F \cap Q_j - \mathrm{cl}U \neq \phi$, for each $F \in \mathcal{F}$ and each $U \in SO_{ij}(x)$. Then $Q_j - \mathrm{cl}U \in \Omega$ for all $U \in SO_{ij}(x)$ and hence Ω $ij - \theta_s$–converges to x. Again, $f(G) \cap P_i - \mathrm{cl}V \neq \phi$, for all $G \in \Omega$ and for all P_j-open nbd V of $y \Rightarrow y \in ji - \theta - \mathrm{cl}f(G)$, for all $G \in \Omega \Rightarrow y \in \cap\{ji - \theta - \mathrm{cl}f(G) : G \in \Omega\}$.

(c) $\Rightarrow$ (a): Let Ω be a grill on X such that $\Omega \xrightarrow{ij-\theta s} x$, and $y \in \cap\{ji - \theta - \mathrm{cl}f(G) : G \in \Omega\}$. Then $\{Q_j - \mathrm{cl}U : u \in SO_{ij}(x)\} \subseteq \Omega$ and $y \in ji - \theta - \mathrm{cl}f(G)$, for each $G \in \Omega$. In particular, $y \in ji - \theta - \mathrm{cl}f(Q_j - \mathrm{cl}U)$ for all $U \in SO_{ij}(x)$. Thus $y \in S_{ij}(f, x)$.

As generalization of Hausdorffness of a topological space, Kelly [6] defined a bitopological space (X, Q_1, Q_2) to be pairwise Hausdorff if for every pair of distinct points x, y of X, there exist a Q_1-open nbd U of x and a Q_2-open nbd V of y such that $U \cap V = \phi$. We now propose to characterize the pairwise Hausdorffness of a space (Y, P_1, P_2) in terms of our introduced concept of ij–S–cluster set of a suitable function f from a space X to Y.

Theorem 2.4. *Let f be a surjection from a space X onto a space Y. Then Y is pairwise Hausdorff if $\cap\{ji - \theta - clf(U) : U \in SO_{ij}(x)\} = \underline{S}_{ij}(f, x)(say) = \{f(x)\}$, for each $x \in X$ and for $i, j = 1, 2 (i \neq j)$.*

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. As f is a surjection, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. As $\underline{S}_{ij}(f, x_1) = \{f(x_1)\}$, $y_2 = f(x_2) \notin \underline{S}_{ij}(f, x_1)$ so that there exists a P_j – open nbd V of y_2 such that $P_i - \mathrm{cl}V \cap f(U) = \phi$, for some $U \in S_{ij}(x_1)$. Then $(Y - P_i - \mathrm{cl}V)(\supseteq f(U))$ is a P_i-open nbd of $f(x_1) = y_1$ and V is a P_j-open nbd of y_2 such that $V \cap (Y - P_i - \mathrm{cl}V) = \phi$. Hence Y is pairwise Hausdorff.

Corollary 2.5. *If $S_{ij}(f, x) = \{f(x)\}$ for $i, j = 1, 2 (i \neq j)$ and for each $x \in X$, where $f : X \rightarrow Y$ is a surjection, then Y is a pairwise Hausdorff space.*

The following example shows that the converse of Corollary 2.5 is false.

Example 2.6. On the set R of real numbers, let $Q_1 = \tau_u$, $Q_2 = \tau_f$, $P_1 = \tau_l$, $P_2 = \tau_r$, where

τ_u, τ_f, τ_l, τ_r respectively denote the usual topology, cofinite topology, lower limit and upper limit topology on R. Then (R, P_1, P_2) is clearly pairwise Hausdorff. Also for any $x \in R$ and any $U \in S_{12}(x)$, $Q_2 - \mathrm{cl}U = R$ and hence $S_{12}(f, x)$, where $f : (R, Q_1, Q_2) \rightarrow (R, P_1, P_2)$ is the identity map, is X. We note in this connection that P_1 and P_2 could be taken to be any topologies on R such that (R, P_1, P_2) is pairwise Hausdorff.

In order to find the condition under which the converse of Corollary 2.5 may hold, we set the following definition.

Definition 2.7. A function $f : X \rightarrow Y$ is called *$ij - \theta_s -$ irresolute* on X if for each $x \in X$ and each ij-semiopen set V in Y containing $f(x)$, there is an ij-semiopen set U in X containing x such that $f(Q_j - \mathrm{cl}U) \subseteq V$.

Theorem 2.8. *Let $f : X \rightarrow Y$ be an $ij - \theta_s -$ irresolute function with Y a pairwise Hausdorff space. Then $S_{ij}(f, x) = \{f(x)\}$, for each $x \in X$.*

Proof. Let $x \in X$. As f is $ij - \theta_s -$ irresolute on X, for any $V \in SO_{ij}(f(x))$ there is $U \in SO_{ij}(x)$ such that $f(Q_j\text{–cl}U) \subseteq V$. Then $S_{ij}(f, x) = \cap\{ji\text{–}\theta\text{–cl}f(Q_j - \mathrm{cl}U) : U \in SO_{ij}(x)\} \subseteq \cap\{ji\text{–}\theta\text{–cl}V : V \in SO_{ij}(f(x))\}$. Let $y \in Y$ with $y \neq f(x)$. As Y is pairwise Hausdorff, there exists a P_j-open nbd U of y and a P_i-open nbd W of $f(x)$ such that $U \cap W = \phi$. Then $y \notin P_j - \mathrm{cl}W = ji - \theta - \mathrm{cl}W$ and $W \in SO_{ij}(f(x))$. Thus $y \notin \cap \{ji - \theta - \mathrm{cl}V : V \in SO_{ij}(f(x))\}$ so that $y \notin S_{ij}(f, x)$. Hence $S_{ij}(f, x) = \{f(x)\}$.

Combining Corollary 2.5 and Theorem 2.8 we get our desired result:

Theorem 2.9. *A space Y is pairwise Hausdorff iff for a space X and an $ij - \theta_s -$ irresolute function $f : X \rightarrow Y$, one has $S_{ij}(f, x) = \{f(x)\}$, for each $x \in X$.*

we have demonstrated above that degeneracy of cluster set of a certain class of functions characterizes Hausdorffness of a bitopological space. It is thus our aim now in the next couple of results to ascertain a few other situations where such degeneracy of cluster sets takes place. We begin with the following definitions.

Definition 2.10. A function $f : X \rightarrow Y$ is called *$ij - \theta -$ closed* if image of any ij–θ–closed set in X is ij–θ–closed in Y.

Definition 2.11. [13] A bitopological space (X, Q_1, Q_2) is called *ij-almost regular* if for each point $x \in X$ and each ij-regular open set G containing x, there is an ij-regular open set H containing x such that Q_j–cl$H \subseteq G$, or equivalently if for each $A \subseteq X$, ij–θ–cl$A$ is ij–θ–closed [12].

Theorem 2.12. *Let f be a ji–θ–closed function from a ji–almost regular space X into a space Y. If $f^{-1}(y)$ is ij–θ–closed in X for all $y \in Y$, then $S_{ij}(f, x) = \{f(x)\}$, for each $x \in X$.*

Proof. We have $S_{ij}(f, x) = \cap\{ji - \theta - \mathrm{cl}f(Q_j - \mathrm{cl}U) : U \in SO_{ij}(x)\} \subseteq \cap\{ji - \theta - \mathrm{cl}f(ji - \theta -$

$\mathrm{cl}U)$: $U \in SO_{ij}(x)\}$. As X is ji-almost regular, $ji-\theta-\mathrm{cl}U$ is $ji-\theta$–closed for all $U \in SO_{ij}(x)$. Now, f being $ji-\theta$–closed, $ji-\theta-\mathrm{cl}f(ji-\theta-\mathrm{cl}U) = f(ji-\theta-\mathrm{cl}U)$ so that $S_{ij}(f, x) \subseteq \cap \{f(ji-\theta-\mathrm{cl}U : U \in SO_{ij}(x)\}$. Now, let $y \in Y$ such that $y \neq f(x), f^{-1}(y)$ being $ij-\theta$–closed and $x \notin f^{-1}(y)$, there exists a Q_i-open nbd V of x such that $Q_i - \mathrm{cl}V \cap f^{-1}(y) = \phi$. So $y \notin f(Q_j - \mathrm{cl}V) = f(ji-\theta-\mathrm{cl}V)$ (since $V \in Q_i$) and hence $y \notin \cap \{f(ji-\theta-\mathrm{cl}U) : U \in SO_{ij}(x)\} \Rightarrow y \notin S_{ij}(f, x) \Rightarrow S_{ij}(f, x) = \{f(x)\}$.

Theorem 2.13. *Let f be a $ji-\theta$–closed injection on a ji-almost regular pairwise Hausdorff space X into Y. Then $S_{ij}(f, x)$ is degenerate for each $x \in X$.*

Proof. As X is ji-almost regular and f is a ji-θ-closed map, we have ji-θ-$\mathrm{cl}f(ji - \theta - \mathrm{cl}U) = f(ji - \theta - \mathrm{cl}U)$, for $U \in SO_{ij}(x)$ and hence for any $x \in X$, $S_{ij}(f, x) = \cap\{ji - \theta - \mathrm{cl}f(Q_j - \mathrm{cl}U) : U \in SO_{ij}(x)\} \subseteq \cap \{ji - \theta - \mathrm{cl}f(ji - \theta - \mathrm{cl}U) : U \in SO_{ij}(x)\} = \cap\{f(ji - \theta - \mathrm{cl}U) : U \in SO_{ij}(x)\}$. For $x_1 \in X$ with $x_1 \neq x$, $f(x_1) \neq f(x)$ as f is injective. By pairwise Hausdorffness of X, there are a Q_i-open nbd U of x and a Q_j-open nbd V of x_1 such that $U \cap Q_i - \mathrm{cl}V = \phi$ so that $f(x_1) \notin S_{ij}(f, x)$. Hence $S_{ij}(f, x) = \{f(x)\}$.

Theorem 2.14. *Let $f : X \to Y$ be a function with $ij-\theta$–closed graph. Then $S_{ij}(f, x)$ is degenerate, for each $x \in X$.*

Proof. For any $y \in S_{ij}(f, x)$, let G be any T_i–basic open nbd of (x, y). Then $G = U \times W$, where $x \in U \in Q_i$ and $y \in W \in P_j$. Now $U \in SO_{ij}(x)$ and hence $y \in ji-\theta-\mathrm{cl}f(Q_j - \mathrm{cl}U) \Rightarrow P_i - \mathrm{cl}W \cap f(Q_j - \mathrm{cl}U) \neq \phi \Rightarrow (Q_j - \mathrm{cl}U \times P_i - \mathrm{cl}W) \cap G(f) \neq \phi$, where $G(f)$ denotes the graph of $f \Rightarrow T_j - \mathrm{cl}(U \times W) \cap G(f) \neq \phi \Rightarrow (x, y) \in ij - \theta - \mathrm{cl}G(f) = G(f) \Rightarrow y \in p_2((\{x\} \times Y) \cap G(f))$, where p_2: $(X \times Y, T_1, T_2) \to (Y, P_1, P_2)$ denotes the projection map $\Rightarrow y = f(x)$. Thus $S_{ij}(f, x) \subset \{f(x)\}$. As $\{f(x)\} \subseteq S_{ij}(f, x)$ is obvious, we have $S_{ij}(f, x) = \{f(x)\}$.

As a partial converse of the above theorem, we have:

Theorem 2.15. *Let $f : X \to Y$ be a function such that $S_{ij}(f, x)$ is degenerate for each $x \in X$. Then the graph $G(f)$ of f is $ij-\theta_s$–closed.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$, so that $y \neq f(x)$ and hence $y \notin S_{ij}(f, x)$. Thus $y \notin ji-\theta-\mathrm{cl}f(Q_j-\mathrm{cl}U)$, for some $U \in SO_{ij}(x)$. This means that for some P_j-open nbd V of y, $P_i - \mathrm{cl}V \cap f(Q_j - \mathrm{cl}U) = \phi$, i.e., $(Q_j - \mathrm{cl}U \times P_i - \mathrm{cl}V) \cap G(f) = \phi$. Hence $T_j - \mathrm{cl}(U \times V) \cap G(f) = \phi$, where $U \times V$ is clearly an ij-semiopen set containing (x, y). Then $(x, y) \notin ij - \theta_s - \mathrm{cl}G(f)$ and $G(f)$ becomes $ij - \theta_s -$ closed.

3. *S*-CLOSEDNESS VIA *S*-CLUSTER SETS

The concept of S-closed topological spaces, first introduced by Thompson [14], has so far been investigated in great detail by many mathematicians. The concept was generalised to bitopological structures in [7]. We recall the definition of bitopological S-closedness below and prove a few results concerning the notion as prerequisites to ultimately characterize it in terms of the concept of ij-S-cluster sets, studied in the preceding section.

Definition 3.1. [7] A subset A of a bitopological space (X, Q_1, Q_2) is called *ij–S–closed relative to X* if for every cover $\mathcal{U}$ of A by ij-semiopen sets of X, there is a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{Q_j - \mathrm{cl}U : U \in \mathcal{U}_0\}$ is again a cover of A. If, in addition, $A = X$, then X is called an *ij–S–closed space.* The space (X, Q_1, Q_2) is said to be *pairwise S-closed* if it is 12–S–closed as well as 21–S–closed.

Theorem 3.2. *A set A in a space X is ij–S–closed relative to X iff for every filterbase $\mathfrak{F}$ on X with $F \cap C \neq \phi$ for each $F \in \mathfrak{F}$ and each $C \in SO_{ij}(A)$, it follows that $A \cap ij - \theta_s - \mathrm{ad}\mathfrak{F} \neq \phi$.*

Proof. Let $A (\subseteq X)$ be ij–S–closed relative to X and $\mathfrak{F}$ be a filterbase on X with the given property. Suppose, if possible, $A \cap ij - \theta_s - \mathrm{ad}\mathfrak{F} = \phi$. Then for each $x \in A$, there is an ij-semiopen set V_x in X containing x such that $Q_j - \mathrm{cl}V_x \cap F_x = \phi$ for some $F_x \in \mathfrak{F}$. Now, $\{V_x : x \in A\}$ is a cover of A by ij-semiopen sets of X. Since A is ij-S-closed relative to X, there exists a finite subset A^* of A such that $A \subseteq \cup \{Q_j - \mathrm{cl}V_x : x \in A^*\}$. Now choose $F^* \in \mathfrak{F}$ such that $F^* \subseteq \cap\{F_x : x \in A^*\}$. Then $F^* \cap (\cup\{Q_j - \mathrm{cl}V_x : x \in A^*\}) = \phi$, i.e., $F^* \cap Q_j - \mathrm{cl}(\cup\{V_x : x \in A^*\}) = \phi$. As $\cup\{V_x : x \in A^*\}$ is an ij-semiopen set in X, so is $Q_j - \mathrm{cl}(\cup\{V_x : x \in A^*\})$, and moreover, $A \subseteq Q_j - \mathrm{cl}(\cup\{V_x : x \in A^*\})$. Thus $Q_j - \mathrm{cl}(\cup\{V_x : x \in A^*\}) \in SO_{ij}(A)$, a contradiction to the hypothesis.

Conversely, let A be not ij–S–closed relative to X. Then for some cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by ij-semiopen set of X, $A \not\subset \bigcup\limits_{\alpha \in \Lambda_0} Q_j - \mathrm{cl}V_\alpha$ for each finite subset Λ_0 of Λ. So $\mathfrak{F} = \{A \setminus \bigcup\limits_{\alpha \in \Lambda_0} Q_j - \mathrm{cl}V_\alpha : \Lambda_0$ is a finite subset of $\Lambda\}$ is a filterbase on X with $F \cap C \neq \phi$ for each $F \in \mathfrak{F}$ and each $C \in SO_{ij}(A)$. But $A \cap ij - \theta_s - \mathrm{ad}\mathfrak{F} = \phi$.

Theorem 3.3. [7] *A space X is if-S-closed iff every filterbase ij-θ_s-adheres in X.*

Lemma 3.4. *Any ij-θ_s-closed subset of an ij-S-closed space X is ij-S-closed relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of an ij–θ_s–closed subset A of X by ij—semiopen sets of X. Now, for each $y \in x \setminus A$, there exists an ij-semiopen set V_y in X such that $y \in V_y$ and $Q_j - \mathrm{cl}V_y \cap A = \phi$. Then $\mathcal{U} \cup \{V_y : y \in X \setminus A\}$ is a cover of X by ij-semiopen sets of X. By ij-S-closedness of X, we obtain a finite subset Λ_0 of Λ and a finite subset $\{y_1, \ldots, y_k\}$ of $(X \setminus A)$ such that $X = (\bigcup\limits_{\alpha \in \Lambda_0} Q_j - \mathrm{cl}U_\alpha) \cup (\bigcup\limits_{i=1}^{k} V_{y_i})$. Then $A \subseteq \bigcup\limits_{\alpha \in \Lambda_0} Q_j - \mathrm{cl}U_\alpha$ and hence A becomes ij-S-closed relative to X.

We are now in a position to prove the desired characterization of ij-S-closed spaces via ij-S-cluster sets as follows, where we use the notation $S_{ij}(f, A)$ to mean the set $\bigcup_{x \in A} S_{ij}(f, x)$, for any $A \subseteq X$ and any function $f : X \to Y$.

Theorem 3.5. *For a bitopological space* (X, Q_1, Q_2) *the following are equivalent :*

(a) X *is* ij-S-*closed.*

(b) $S_{ij}(f, A) \supseteq \cap \{ji - \theta - \mathrm{cl} f(U) : U \in SO_{ij}(A)\}$ *for each* ij–θ_s–*closed subset* A *of* X, *for each space* (Y, P_1, P_2) *and each function* $f : X \to Y$.

(c) $S_{ij}(f, A) \supseteq \cap \{ji - \theta_s - \mathrm{cl} f(U) : U \in SO_{ij}(A)\}$ *for each* $ij - \theta_s -$ *closed subset* A *of* X, *for each space* (Y, P_1, P_2) *and each function* $f : X \to Y$.

Proof. (a) $\Rightarrow$ (b): Let A be any $ij - \theta_s -$ closed subset of X, where X is ij-S-closed. Then by Lemma 3.4, A is ij-S-closed relative to X. Now, let $z \in \cap \{ji - \theta - \mathrm{cl} f(W) : W \in SO_{ij}(A)\}$. Then for each Q_j-open nbd V of z and for each $W \in SO_{ij}(A)$, $Q_i - \mathrm{cl} V \cap f(W) \neq \phi \Rightarrow f^{-1}(Q_i - \mathrm{cl} V) \cap W \neq \phi$. Thus $\mathfrak{F} = \{f^{-1}(Q_i - \mathrm{cl} V) : z \in V \in Q_j\}$ is a filterbase on X satisfying the condition of Theorem 3.2. Hence there exists $x \in A \cap ij - \theta_s - \mathrm{ad} \mathfrak{F}$. Then $x \in A$, and for all $U \in SO_{ij}(x)$ and each Q_j-open nbd V of z, Q_j-cl $U \cap f^{-1}(Q_i - \mathrm{cl} V) \neq \phi \Rightarrow f(Q_j - \mathrm{cl} U) \cap Q_i - \mathrm{cl} V \neq \phi \Rightarrow z \in ji - \theta - \mathrm{cl} f(Q_j - \mathrm{cl} U)$, for all $U \in SO_{ij}(x) \Rightarrow z \in S_{ij}(f, A)$.

(b) $\Rightarrow$ (c) : It is obvious.

(c) $\Rightarrow$ (a) : For proving X to be ij-S-closed, it is to be shown by virtue of Theorem 3.3 that every filterbase $\mathfrak{F}$ on X, $ij - \theta_s -$ adheres at some $x \in X$. Let $\mathfrak{F}$ be a filterbase on X. Take $y_0 \notin X$(e.g. $y_0 = X$) and $Y = X \cup \{y_0\}$. Then $\tau = \{A \subseteq Y : y_0 \notin A$, or $F \subseteq A$, for some $F \in \mathfrak{F}\}$ is a topology on Y. We put $P_1 = P_2 = \tau$ and consider the identity map $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$. Then $\cap \{ji - \theta_s - \mathrm{cl} f(U) : U \in SO_{ij}(X)\} = \cap \{ji - \theta_s - \mathrm{cl} U : U \in SO_{ij}(X)\} = (ji - \theta_s - \mathrm{cl} X)$ in Y. Now, $y_0 \in ji - \theta_s - \mathrm{cl} X$ in Y. Hence by (c), $y_0 \in S_{ij}(f, X)$. Choose $x \in X$ such that $y_0 \in S_{ij}(f, x)$. Then for all $V \in SO_{ij}(x)$ and any $F \in \mathfrak{F}$, $F \cup \{y_0\}$ is P_j-open as well as P_i-closed and contains y_0 such that $f(Q_j - \mathrm{cl} V) \cap (F \cup \{y_0\}) \neq \phi$. This implies that $Q_j - \mathrm{cl} V \cap F \neq \phi$ for all $V \in SO_{ij}(x)$ so that $x \in ij - \theta_s - \mathrm{cl} F$ for all $F \in \mathfrak{F}$. Thus $X \cap ij - \theta_s - \mathrm{ad}\, \mathfrak{F} \neq \phi$ and consequently X becomes ij–S–closed.

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IDEALS AND RADICALS IN THE TENSOR PRODUCT OF BANACH ALGEBRAS

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ABSTRACT : In this paper, we prove some enlightening results regarding the ideals and radicals in the tensor product of two Banach algebras. We show that in a Banach algebra nil property and the local nilpotence are radical properties. Let A_1 and A_2 be two Banach algebras with identities. Here, we construct the nil radical in the projective tensor product, $A_1 \otimes_\gamma A_2$ provided the nil radicals in A_1 and A_2 are given. Again, given Levitzki radicals in A_1 and A_2, we construct the same for $A_1 \otimes_\gamma A_2$ also. The converse of these results are also established.

Key words : Nil property, Local nilpotence, Projective tensor product, Levitzki radical.

AMS subject Classification (2000): Primary section: 46M05,
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1. INTRODUCTION

Let A_1 and A_2 be two Banach algebras, α be an algebra norm on the algebraic tensor product $A_1 \otimes A_2$ and $A_1 \otimes_\alpha A_2$ be the completion of $A_1 \otimes A_2$ with respect to α. In 1965, Divinsky made a systematic study of all the eight radicals in rings. Dixon [4] in his paper, discussed about three radicals in an algebra. In this paper, we extend these ideas to the projective tensor product of two Banach algebras. Let A_1 and A_2 be two Banach algebras with identities and let the nil radicals in A_1 and A_2 be given. With these conditions, in §2 of our paper, we construct the nil radical in $A_1 \otimes_\gamma A_2$. The converse of this result is also established here. Moreover, we prove analogous results in the case of Levitzki radical in the projective tensor product of two Banach algebras.

We first highlight some notations and terminologies which are useful for our purpose.

Definition 1.01. Let A be a Banach algebra. Let ς be a certain property that an algebra may possess. We say that the algebra A is a ς- algebra if it possesses the property ς.

An ideal J of A is called a ς-ideal if J is a ς-algebra. An algebra which does not contain any non-zero ς-ideal is called ς-semi-simple.

We call ς a *radical property* if the following three conditions hold:

(i) A homomorphic image of a ς-algebra is a ς-algebra.

(ii) Every algebra contains a ς-ideal S which contains every other ς-ideal of the algebra.

(iii) The factor algebra A/S is ς-semi-simple.

The following Theorem gives us conditions which are equivalent to (i), (ii) and (iii).

Theorem 1.02. Property ς is a *radical property* if and only if

(i) A homomorphic image of a ς-algebra is a ς-algebra.

(ii) If every non-zero homomorphic image of an algebra A contains a non-zero ς-ideal then A is a ς-algebra.

The maximal ς-ideal of an algebra A is called the ς-*radical* of A.

Definition 1.03. An element x in an algebra is said to be *nilpotent* if ∃ a positive integer n such that $x^n = 0$. An algebra A is said to be *nil* if every element x in A is nilpotent, i.e., $x^n = 0$, where n depends on the particular element x of A.

Definition 1.04. An algebra is *locally nilpotent* if any finite set of elements generates a subalgebra which is nilpotent.

Thus, every nilpotent algebra is locally nilpotent, and every locally nilpotent algebra is nil.

Algebraic tensor product 1.05. Let X, Y be normed spaces over F with dual spaces X' and Y'. Given $x \in X$, $y \in Y$, let $x \otimes y$ be the element of BL(X', Y'; F) defined by $x \otimes y(f,g) = f(x)g(y)(f \in X', g \in Y')$.

The algebraic tensor product of X and Y, $X \otimes Y$, is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in BL(X', Y'; F).

Cross norm 1.06. A norm $\|\cdot\|_\alpha$ on $X \otimes Y$ is called a *cross norm* if

$$\| x \otimes y \|_\alpha \leq \| x \| \, \| y \| \quad \forall \; x \in X, \; y \in Y.$$

Greatest cross norm or projective tensor norm 1.07. Given normed spaces X, Y, the projective tensor norm $\|\cdot\|_\gamma$ on $X \otimes Y$ is defined as:

$$\|u\|_\gamma = \inf\left\{\sum_i \|x_i\| \, \|y_i\| : u = \sum_i x_i \otimes y_i\right\}$$

where the infimum is taken over all (finite) representations of u.

The completion of $(X \otimes Y, \|\cdot\|_\gamma)$ is called the *projective tensor product* of X and Y, and is denoted by $X \otimes_\gamma Y$.

Least cross norm or injective tensor norm 1.08. Given normed spaces X, Y, the linear space $X \otimes Y$ inherits a natural norm from BL(X', Y'; F). This norm is called the *weak* or *injective tensor norm* and is denoted by

$$\|u\|_\lambda = \sup\left\{\left|\sum_i f(x_i)g(y_i)\right| : \|f\| \leq 1, \|g\| \leq 1\right\} \text{ where } u = \sum_i x_i \otimes y_i$$

The completion of $(X \otimes Y, \|\,.\,\|_\lambda)$ is called the *injective tensor product* of X and Y, and is denoted by $X \otimes_\lambda Y$.

2. MAIN RESULTS

We need to arm with the following Lemmas and Theorems in order to achieve our main results.

Lemma 2.01. If A is a nil algebra, so is every subalgebra of A and so is every homomorphic image of A. If S is an ideal of the algebra A and both A/S and S are nil, then so is A.

[The proof follows immediately from the definitions.]

Lemma 2.02. The sum two nil ideals I_1 and I_2, of an algebra A is again a nil ideal.

Proof: Since $(I_1 + I_2) / I_2 \cong I_1 / (I_1 \cap I_2)$, by the second isomorphism Theorem, and since the RHS is a homomorphic image of the nil ideal I_1, the LHS is a nil ideal by Lemma 2.01. Since both I_2 and $(I_1 + I_2) / I_2$ are nil, using Lemma 2.01, $(I_1 + I_2)$ is nil.

Corollary 2.03. The sum of any finite number of nil ideals of an algebra is again a nil ideal.

[The proof follows by induction.]

Lemma 2.04. The sum of all the nil ideals of an algebra A is a nil ideal.

Proof: Let W be the sum of all the nil ideals of A. If $x \in W$, then x is in some finite sum of nil ideals of A and thus, by the above Corollary, x is nilpotent. Hence, W is nil.

Thus, we have:

Theorem 2.05. *The nil property is a radical property.*

Now, we concentrate on the study of the nil radical in the projective tensor product of two Banach algebras.

Let A_1 and A_2 be two Banach algebras. Let $A_3 = A_1 \otimes_\gamma A_2$, Let $S(A_i)$, i = 1, 2, 3 be the set of *regular maximal ideals* of A_i respectively.

Theorem 2.06. Let A_1 and A_2 be two Banach algebras with identities. Let $A_3 = A_1 \otimes_\gamma A_2$. If the nil radicals in A_1 and A_2 are given, we can construct the nil radical in A_3.

Proof: Let J_1 and J_2 be nil radicals in A_1 and A_2 respectively. So, J_1 and J_2 are the maximal nil ideals in A_1 and A_2 respectively. Let $M_1 \in S(A_1)$ such that $J_1 \subseteq M_1$ and $M_2 \in S(A_2)$ such that $J_2 \subseteq M_2$.

Case 1: *A_1 is commutative and A_2 is non-commutative:*

Since A_1 is commutative, $\exists$ a homomorphism $h: A_1 \to C$ such that $M_1 = \ker h$.

We define $\phi_h: A_3 \to A_2$ defined by

$$\phi_h\left(\sum_i a_{1i} \otimes a_{2i}\right) = \sum_i h(a_{1i})a_{2i} \in A_2.$$

Then ϕ_h is a homomorphism. Let u_1 and u_2 be the identities modulo the ideal M_1 and M_2 respectively. We take $M_3 = \phi_h^{-1}(M_2)$. Clearly, M_3 is a modular ideal with $u_1 \otimes u_2$ as an identity modulo M_3.

Let M'_3 be an ideal in A_3 such that $M'_3 \supset M_3$.

From the definition of M_3, we get $\phi_h(M'_3) \supset M_2$, i.e., $\phi_h(M'_3) = A_2$, since M_2 is maximal. Hence, there is $t \in M'_3$ so that $\phi_h(t) = u_2$. Now, $\phi_h(t - u_1 \otimes u_2) = \phi_h(t) - h(u_1)u_2 = u_1 - u_2 = 0$; so, $t - u_1 \otimes u_2 \in M_3 \subset M'_3$. Hence, $u_1 \otimes u_2 \in M'_3$ and $M'_3 = A_3$. It follows that M_3 is maximal.

Now let $b \in J_2 \subseteq M_2$, since J_2 is nil, so $\exists$ a positive integer n such that $b^n = 0$.

We have, $u_1 \otimes b \in A_3$. $\phi_h(u_1 \otimes b) = h(u_1)b = b \in M_2$. So, $u_1 \otimes b \in M_3$.

Also, $(u_1 \otimes b)^n = u_1{}^n \otimes b^n = 0 \Rightarrow u_1 \otimes b$ is nilpotent.

Hence, M_3 contains at least one nilpotent element. Let J_3 be the sum of all the ideals $<y>$ where $y \in M_3$ is a nilpotent element. Then J_3 is a nil ideal.

Let $J_3 \subset P \subset M_3$, where P is a nil ideal in A_3. Then, $\exists$ $\sum_i a_i \otimes b_i \in P$ such that

$\sum_i a_i \otimes b_i \notin J_3$. Since P is nil, $\left(\sum_i a_i \otimes b_i\right)^n = 0$ for some positive integer n.

$\Rightarrow \sum_i a_i \otimes b_i$ is nilpotent $\Rightarrow \sum_i a_i \otimes b_i \in J_3$, which is a contradiction.

So, J_3 is the maximal nil ideal in A_3 i.e., it is the nil radical in A_3.

Case 2: *A_1 and A_2 are non-commutative:*

We define the linear map:

R: $A_3 \to (A_1/M_1) \otimes_\gamma (A_2/M_2)$ such that if $a_3 = \sum_n a_{1n} \otimes a_{2n}$ then

$$R(a_3) = \sum_n \left({}^{a_{1n}}\!/_{M_1} \right) \otimes \left({}^{a_{2n}}\!/_{M_2} \right).$$

Then Ker R is an ideal in A_3. Let $M_3 \in S(A_3)$ such that ker $R \subset M_3$. Since J_1 is nil, so $\exists\, a \in J_1 \subseteq M_1$ and a positive integer n such that $a^n = 0$; since J_2 is nil, so $\exists\, b \in J_2 \subseteq M_2$ and a positive integer m such that $b^m = 0$.

Now, $a \otimes b \in \ker R$ and $(a \otimes b)^n = a^n \otimes b^n = 0$.

Hence, M_3 contains at least one nilpotent element. Let J_3 be the sum of all the ideals $\langle y \rangle$ where $y \in M_3$ is a nilpotent element. Then J_3 is a nil ideal. As in the previous case, J_3 is the maximal nil ideal in A_3 i.e., it is the nil radical in A_3.

The converse part of the above Theorem can be proved as:

Theorem 2.07. Let A_1 and A_2 be two Banach algebras with identities e_1 and e_2 respectively. Let $A_3 = A_1 \otimes_\gamma A_2$. If the nil radical in A_3 is given, we can construct the same for A_1 and A_2 also.

Proof: Let J_3 be the nil radical in A_3. We take $M_3 \in S(A_3)$ such that $J_3 \subseteq M_3$.

Case 1: *A_1 and A_2 are commutative:*

Let $\pi : A_3 \to A_3/M_3$ be the cononical mapping and let π_i be the restriction of π to A_1 with the help of the identifications: $A_1 \leftrightarrow A_1 \otimes e \subseteq A_3$ and $A_2 \leftrightarrow e \otimes A_2 \subseteq A_3$.

The kernel of π_1 is easily seen to be $A_1 \cap M_3$; so in the diagram below provided us by the induced mapping theorem u: $A_1/A_1 \cap M_3 \to A_3/M_3$ is seen to be an isomorphism.

$$\begin{tikzcd}
A_1 \arrow[rr, "\pi_1"] \arrow[dr, "\sigma"] & & A_3/M_3 \\
& A_1/A_1 \cap M_3 \arrow[ur, "u"'] &
\end{tikzcd}$$

Since A_1 is commutative $\pi_1(A_1)$ is an abelian subalgebra of A_3/M_3. Since $\pi_1(A_1)$ and $\pi_2(A_2)$ commute elementwise, $\pi_1(A_1) \subseteq$ center $(\pi_1(A_1).\pi_2(A_2))$.

Consequently, from $A_1 \otimes A_2$ being dense in $A_1 \otimes_\gamma A_2$, we infer that $\pi_1(A_1) \subseteq$ center

(A_3/M_3). From the diagram, we then conclude $u(A_1/A_1 \cap M_3) = u(\sigma(A_1)) = \pi_1(A_1) \subseteq$ center (A_3/M_3). Therefore, $A_1/A_1 \cap M_3 \cong C$ because A_3/M_3 is primitive. It follows that $A_1 \cap M_3$ is a maximal ideal of A_1.

Let J_1 be the sum of all the ideals $<y>$ where $y \in A_1 \cap M_3$ is a nilpotent element.

We assume that P be a nil ideal in A_1 containing J_1. Then, $\exists$ $a \in P$ such that $a \notin J_1$.

Since P is nil, so $\exists$ a positive integer n such that $a^n = 0$; hence, a is a nilpotent element in $A_1 \cap M_3$. So, $a \in J_1$, which is a contradiction. So, our assumption is wrong. Hence, J_1 is the maximal nil ideal in A_1 i.e., it is the nil radical in A_1. Similarly, we can construct a nil radical J_2 in A_2.

Case 2: *A_1 and A_2 are non-commutative:*

We consider a complex homomorphism ϕ on A_1. Then, we define

$f_\phi : M_3 \to A_2$ such that

$$f_\phi\left(\sum_i a_{1i} \otimes a_{2i}\right) = \sum_i \Phi(a_{1i})a_{2i} \in A_2.$$

Then, f_ϕ is a homomorphism. $f_\phi(M_3) = M_2$(say) is an ideal in A_2. Let P be an ideal in A_2 such that $M_2 \subset P \subset A_2$. So, $\exists$ $x \in P$ such that $x \notin M_2 = f_\phi(M_3)$. Hence, $\exists$ no $\sum_i a_{1i} \otimes a_{2i} \in$ M_3 such that $f_\phi\left(\sum_i a_{1i} \otimes a_{2i}\right) = x \Rightarrow \sum_i \Phi(a_{1i})a_{2i} = x.$

Now, $e_1 \in A_1$, $x \in A_2 \Rightarrow e_1 \otimes x \in A_3$.

$e_1 \otimes x \notin M_3$ since $f_\phi(e_1 \otimes x) = \phi(e_1)x = x \notin f_\phi(M_3)$.

Let $P_1 = <e_1 \otimes x, M_3>$. Then, $M_3 \subset P_1$, which is a contradiction, since M_3 is a maximal ideal in A_3. So, M_2 is a maximal ideal in A_2.

Since J_3 is nil, $\exists$ $\sum_i a_{1i} \otimes a_{2i} \in J_3 \subset M_3$ such that $\left(\sum_i a_{1i} \otimes a_{2i}\right)^n = 0$ for some positive integer n. $f_\phi\left(\sum_i a_{1i} \otimes a_{2i}\right) = \sum_i \Phi(a_{1i})a_{2i} \in M_2$.

Now, $f_\phi\left(\sum_i a_{1i} \otimes a_{2i}\right)^n = 0.$

$$\Rightarrow \left(f_\phi\left(\sum_i a_{1i} \otimes a_{2i}\right)\right)^n = 0.$$

$$\Rightarrow \left(\sum_i \Phi(a_{1i})a_{2i}\right)^n = 0.$$

$$\Rightarrow \sum_i \Phi(a_{1i})a_{2i} \text{ is nilpotent.}$$

So, M_2 contains at least one nilpotent element. Let J_2 be the sum of all the ideals <y> where $y \in M_2$ is a nilpotent element.

Let $J_2 \subset J'_2 \subset A_2$, where J'_2 is a nil ideal in A_2. Then, $\exists$ $a \in J'_2$ such that $a \notin J_2$.

Since J'_2 is nil, so $\exists$ a positive integer n such that $a^n \doteq 0$. Clearly, $a \notin M_2$.

Let $M'_2 = < a, M_2>$. Then, $M_2 \subset M'_2$, which is a contradiction, since M_2 is a maximal ideal in A_2. So, J_2 is the maximal nil ideal in A_2 i.e., it is the nil radical in A_2. Similarly, we can construct a nil radical J_1 in A_1.

Our next aim is to study the Levitzki radical in the tensor product. Here, we use the following simple but illuminating Lemmas:

Lemma 2.08. If an ideal I of an algebra A is locally nilpotent and if A/I is locally nilpotent, then A itself is locally nilpotent. Furthermore, if A is locally nilpotent, then so is every subalgebra and every homomorphic image of A.

Lemma 2.09. The sum of two locally nilpotent ideals I_1 and I_2 of an algebra A is again a locally nilpotent ideal.

Corollary 2.10. The sum of any finite number of locally nilpotent ideals is a locally nilpotent ideal.

Lemma 2.11. The union L of all the locally nilpotent ideals of an algebra A is a locally nilpotent ideal.

Lemma 2.12. A/L has no locally nilpotent non-zero ideals.

[For the proof of these Lemmas, we refer to [3].]

With the help of these Lemmas, we have:

Theorem 2.13. *Local nilpotence is a radical property.*

We may say, then, that this radical property is the *lower radical property* determined by all locally nilpotent algebras.

The maximal locally nilpotent ideal of an algebra is called the *Levitzki radical.*

The following Theorems and Lemmas are technically very powerful and plays a crucial role in establishing our result regarding the Levitzki radical and the tensor product of two Banach algebras A_1 and A_2.

Theorem 2.14 [1]. Let A be a Banach algebra, let J be a proper modular ideal of A, and let u be a modular unit for J. Then

(i) $\|u - x\| \geq 1$ $(x \in J)$;

(ii) the closure of J is a proper modular ideal of A.

Corollary 2.15. Each maximal modular ideal of a Banach algebra is closed.

Lemma 2.16 [5]. Let A be a Banach algebra. Let $n \in N$ and let $x^n = 0$ $(x \in A)$. Then $A^m = \{0\}$, where $m = 2^n - 1$.

Theorem 2.17. Every nil Banach algebra is nilpotent.

Proof: Let A be a nil Banach algebra and, for $n \in N$, let $A_n = \{x \in A : x^n = 0\}$. Then, A_n is closed and $A = \cup\{A_n : n \in N\}$. By the Baire category Theorem, $\exists$ $m \in N$, $b \in A$, $r > 0$ such that $\{a : \|a - b\| < r\} \subset A_m$.

Given $x \in A$, $\exists\delta > 0$ such that $b + \lambda x \in A_m(|\lambda| < \delta)$; and so, $(b + \lambda x)^m = 0(|\lambda| < \delta)$, $x^m = 0$. Now, applying Lemma 2.16, we get that A is nilpotent.

Now, we are in a position to prove:

Theorem 2.18. Let A_1 and A_2 be two Banach algebras with identities. Let $A_3 = A_1 \otimes_\gamma A_2$. If the Levitzki radicals in A_1 and A_2 are given, we can construct the Levitzki radical in A_3.

Proof: Let L_1 and L_2 be the Levitzki radicals in A_1 and A_2 respectively. So, L_1 and L_2 are the maximal locally nilpotent ideals in A_1 and A_2 respectively.

L_1 is locally nilpotent $\Rightarrow$ L_1 is nil.

L_2 is locally nilpotent $\Rightarrow$ L_2 is nil.

Let J_1 be the nil radical in A_1. So, $L_1 \subseteq J_1$.

Let J_2 be the nil radical in A_2. So, $L_2 \subseteq J_2$.

Now, we can construct a nil radical J_3 for $A_3 = A_1 \otimes_\gamma A_2$.

Let $J_3 \subset P \subset A_3$, where P is a locally nilpotent ideal. P is a locally nilpotent $\Rightarrow$ P is nil, which is impossible as J_3 is the maximal nil ideal in A_3.

So, J_3 is not contained in any other locally nilpotent ideal.

By the construction, J_3 = sum of all the nilpotent ideals <y> where y is a nilpotent element in M_3, the regular maximal ideal in A_3 containing J_3.

Now, M_3 is a maximal regular ideal in A_3. Hence, M_3 is closed (by Corollary 2.15). So, $M_3 = \mathrm{cl}(M_3)$ [cl () denotes the closure].

Let $x \in \mathrm{cl}(J_3)$ such that $x = \lim x_n$, $x_n \in J_3$. Clearly, $x \in M_3$.

Since J_3 is nil, $x_n{}^k = 0$ for some $k \in N$.

So, $x^k = 0 \Rightarrow x$ is a nilpotent element in M_3

$\Rightarrow x \in J_3$.

Thus $J_3 = \mathrm{cl}(J_3)$. Hence, J_3 is closed.

So, J_3 is a closed subalgebra of the Banach algebra A_3 Therefore, J_3 is a nil Banach algebra. By Theorem 2.17, J_3 is nilpotent. This implies that J_3 is locally nilpotent.

Hence, we get,

J_3 is the locally nilpotent ideal in A_3 which is not contained in any other locally nilpotent ideal. Therefore, J_3 is the maximal locally nilpotent ideal in A_3. So, J_3 is the Levitzki radical in A_3.

We aim at proving the converse part of the above result by the following:

Theorem 2.19. Let A_1 and A_2 be two Banach algebras with identities e_1 and e_2 respectively. Let $A_3 = A_1 \otimes_\gamma A_2$. If the Levitzki radical in A_3 is given, we can construct the same for A_1 and A_2 also.

Proof: Let L_3 be the Levitzki radical in A_3.

So, L_3 is the maximal locally nilpotent ideal in A_3.

L_3 is locally nilpotent $\Rightarrow L_3$ is nil.

Let J_3 be the nil radical in A_3. So, $L_3 \subseteq J_3$.

Now, we can construct nil radicals J_1 and J_2 in A_1 and A_2 respectively. Each maximal ideal in A_1 and A_2 are regular and hence closed. So, the maximal ideals M_1 and M_2 corresponding to J_1 and J_2 are also closed. Then we get, J_1 and J_2 are also closed (as in the previous Theorem). Hence, J_1 is a closed subalgebra of the Banach algebra A_1. Therefore, J_1 is a nil Banach algebra. By Theorem 2.17, J_1 is nilpotent. This implies that J_1 is locally nilpotent.

Let $J_1 \subset P \subset A_1$, where P is a locally nilpotent ideal in A_1. P is a locally nilpotent $\Rightarrow$ P is nil, which is impossible as J_1 is the maximal nil ideal in A_1.

Thus, J_1 is the maximal locally nilpotent ideal in A_1. Therefore, J_1 is the Levitzki radical in A_1. Similarly, we can show that J_2 is the Levitzki radical in A_2.

3. CONCLUSION

Throughout this paper, we consider only the projective tensor product of two Banach algebras. But regarding injective tensor norm, we cannot apply these results to the injective tensor product of two Banach algebras since for most algebras the injective tensor norm is not submultiplicative. Again, we have proved the results considering the two Banach algebras A_1 and A_2 as having identities. Thus, we can pose the following problems:

1) Can we establish similar results if the two Banach algebras A_1 and A_2 are without identity?
2) What can be said about the nil radical and Levitzki radical in the injective tensor product of two Banach algebras?

   In his paper, Dixon stated the existence of Baer lower radical in any algebra A and characterized it as a limit of a certain transfinite sequence of ideals B_λ. [For details, we refer to [4].] In this context, another interesting problem can be raised as:

3) What will be the structure of Baer lower radical in $A_1 \otimes_\gamma A_2$ if the structure of the same for A_1 and A_2 are given?

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SIMILARITY ORBITS AND FINITE OPERATORS

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ABSTRACT : Let $\mathcal{A}$ denote a complex Bannach algebra with identity e and let $\mathfrak{F}$ be the set of all finite elements (in the sense of J.P. Williams) of $\mathcal{A}$. In this paper we introduce some operators under which $\mathfrak{F}$ is not invariant under similarity orbit.

1. INTRODUCTION

Let $B(H)$ be the algebra of all bounded operators acting on a complex Hibert space $\mathcal{H}$. For A and B in $B(H)$, let $\delta_{A,B}$ denote the operator on $B(H)$ defined by $\delta_{A,B}(X) = AX - XB$. If $A = B$, then δ_A is called the inner derivation induced by A. Let $\mathcal{A}$ denote a complex Banach algebra with identity e. A state on $\mathcal{A}$ is a functional $f \in \mathcal{A}^*$ such that $f(e) = 1 = \|f\|$. For $x \in \mathcal{A}$, let

$$W_0(x) = \{f(x) : f \text{ is a state on } \mathcal{A}\},$$

the numerical range of x [9]. Then $W_0(x)$ is a compact convex set containing $co\sigma(x)$ (the convex hull of the spectrum of x) [9]. For the case $\mathcal{A} = B(H)$, if $A \in B(H)$, then $W_0(A) = \overline{W(A)}$, where

$$W(A) = \{(Ah, h) : h \in \mathcal{H}, \|h\| = 1\}$$

(the spatial unmerical range of A). An element a is finite if

$$0 \in W_0(ax - xa)$$

for each $x \in \mathcal{A}$: $\mathfrak{F}(A)$ (or $\mathfrak{F}$) denotes the set of all finite elements of $\mathcal{A}$. J.P. Williams proved that $\mathfrak{F}$ contains every normal and hyponormal operators. In [6] we proved that $\mathfrak{F}$ contains a more general classes of operators. D.A. Herrero and Fialkow [2] asserts that $\mathfrak{F}(H)$ is not invariant under compact perturbation. In this paper we show that $\mathfrak{F}$ is not invariant under similarity orbits.

2. SIMILARITY ORBITS

Let $S(A) \equiv \{X^{-1}AX : X \in B(H)\}$, where X is invertible be the similarity orbit of

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$A \in B(H)$ and let $SF(H)$ denote the union of all similarity orbits $S(A)$, where A ranges through the class of finite operators. In this paper we show that $\mathfrak{F}(H) \not\subset S\mathfrak{F}(H)$.

Let $T(A)$ be the operator defined on $B(K)$ by

$$T(A) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ A & I & 0 \end{bmatrix},$$

where

$$K = H^{(3)} = H \oplus H \oplus H.$$

It is well known [9. Theorem 8] that $T(A)$ is finite if and only if A is finite.

Theorem 2.1. *The operator defined on $B(K)$ by*

$$T(0) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

where

$$K = H^{(3)} = H \oplus H \oplus H$$

is similar to some operator, which is not in $\mathfrak{F}(H)$.

Proof : It suffices to show the existence of an operator similar to $T(A)$, for some $A \in \mathfrak{F}(H)$, which is not in $\mathfrak{F}(H)$. Let

$$S(A) = \begin{bmatrix} I & 0 & 0 \\ -A & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

then $[S(A)]^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A & 0 & I \end{bmatrix}$

and

$$S^{-1}T(A)S = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} = T(0).$$

Now from $T(0)$, we construct a similar operator by using $S(B)$ for an arbitrary B in $B(H)$, since

$$T(B) = [S(B)D(0)S(B)]^{-1}.$$

if A was in $S\mathfrak{F}(H)$, then $B(H) = \mathfrak{F}(H)$ by [9, Theorem 8] which is impossible.

It is known that, if $a, x \in \mathcal{A}$ and $A, X \in B(H)$, the $W_0(ax-xa)$ is the closure $\overline{W(AX - XA)}$ of the numerical range in the usual sense in Hilbert space, given by

$$W(B) = \{(Bx \mid x), \|x\| = 1\}.$$

It is well known that if $co\sigma(B)$ is the convex hull of the spectrum of B, then

$$co\sigma(B) = \bigcap \left\{\overline{W(SBS^{-1})},\ S \in B(H)\right\}$$

where S is invertible. Hence

$$SF(H) = \{B \in B(H) : 0 \in co\sigma\ (B)\}.$$

Recall that the rank-one operator $e \otimes f$ is defined by

$$(c \otimes f)g = (f,g)\ e.$$

In the following part $T(A)$ will be represented, by

$$T(A) = \begin{bmatrix} 0 & I & A \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

on $K = H \oplus H \oplus H$.

Theorem 2.2. Let $G = C \oplus K$ *and* $M = M\ (A) = 0 \oplus T(A)$. *Then* M *is similar to some operator, which is not finite.*

Proof : We will construct $Z \in B(G)$ such that

$$0 \notin co\sigma\ (MZ - ZM).$$

Since $W(.)$ is a convex subset and

$$W\ (\exp\ (i\alpha)\ (TY - YT)) = \exp(i\alpha)W(TY - YT).$$

We choose Y by the choice of a scalar factor of modulus 1 such that

$$\overline{W(TY-YT)} \subset R^2_+ = \{z \in C : \text{Re}z > 0\}.$$

Let Z be defined on

$$G = C \oplus K$$

by $Z = \begin{bmatrix} 0 & C \\ D & Y \end{bmatrix}$

Then $MZ - ZM = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}\begin{bmatrix} 0 & C \\ D & Y \end{bmatrix} - \begin{bmatrix} 0 & C \\ D & Y \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0 & CT \\ TD & TY - YT \end{bmatrix}$

for the vectors $e = 1$ of C, f and g are two vectors of K, $C \in B(K,C)$ and $D \in B(C,K)$ are associated by

$$C = e \otimes f$$

and

$$D = g \otimes e.$$

Then

$$MZ - ZM = \begin{bmatrix} 0 & e \otimes T^* f \\ Tg \otimes e & TY - YT \end{bmatrix}$$

Denote

$$R = TY - YT.$$

We will show a suitable choice of f,g which gives Z the desired property. With respect to the following decomposition

$$K = H^{(3)} = H \oplus H \oplus H,$$

$$f = (f_1, f_2, f_3), \; g = (g_1, g_2, g_3)$$

we have

$$Tg = \begin{bmatrix} 0 & I & A \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} g_2 + Ag_3 \\ g_3 \\ 0 \end{bmatrix};$$

$$T^* f = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ A^* & I & 0 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ A^* f_1 + f_2 \end{bmatrix}.$$

Choose f and g by

$$A^* f_1 + f_2 = 0, \; g_2 + Ag_3 = 0,$$

$$g_3 = f_1 = 0.$$

If we put

$$h = Tg = T^* f,$$

then

$$MZ - ZM = \begin{bmatrix} 0 & e\otimes h \\ h\otimes e & R \end{bmatrix}$$

If

$$x = \alpha e + y,\ \alpha \in C,\ y \in K$$

and for $\nu \in C$, we have $Mx = \nu x$, then

$$\begin{bmatrix} 0 & e\otimes h \\ h\otimes e & R \end{bmatrix}\begin{bmatrix} \alpha \\ y \end{bmatrix} = \begin{bmatrix} (y|h) \\ \alpha h + Ry \end{bmatrix} = \nu\begin{bmatrix} \alpha \\ y \end{bmatrix}$$

A solution x satisfies

$$(y \mid h) = \nu\alpha$$

$$\alpha h = -R_\nu y$$

(where $R_\nu = R - \nu I$). If R_ν is invertible, then

$$\nu\alpha = (R_\nu y \mid [R_\nu^{+}]^{-1}h) = -\alpha(h \mid [R^{+}_{\nu}]^{-1}h).$$

If x is a non-zero solution, then $\alpha \neq 0$ and $y \neq 0$, hence ν does satisfy

$$\nu = (-[R_\nu]^{-1}h]\ h).$$

for any possible values of ν. By the choice of

$$\overline{W(TY-YT)} \subset R^2_+,$$

$R = TY - YT$ has a spectral set Δ (a closed disk included in R^2_+). If $Re\nu < 0$, the map defined on R by

$$\phi(z) = (z - \nu)^{-1}$$

which has a pole out of Δ, has for spectral set $\phi(\Delta)$ which is included in C/R^2_+. Since every convex spectral set contains the adherence of the numerical range, we have

$$R_e\,([R_\nu]^{-1}h \mid h) < 0.$$

Then $Re(\nu) > 0$. If $\nu \in \Sigma = \{z \in C : Rez \leq 0\}$, then R_ν is invertible and M_ν is injective. We will show that M_ν is also surjective. Let

$$x' = \alpha' e + y'$$

defined on $G = C \oplus K$, and

$$M_\nu x = x'$$

can be written as follows :

$$(y \mid h) - v\alpha = \alpha',$$

$$\alpha h + R_v y = y'.$$

This implies,

$$\alpha[v + ([R_v]^{-1}h \mid h) + \alpha' = (y' \mid [R_v^*]^{-1}h).$$

Since

$$[v + ([R_v]^{-1}h \mid h)]$$

is strictly negative, this gives the values of α, y. Using the fact that

$$y = [R_v]^{-1}(y' - \alpha h),$$

we deduce that M_v is surjective. By the above arguments no points of the spectrum of M are in Σ and $\sigma(M) \subset \mathbb{R}^2_+$. Since it is a closed set, we also have $co\sigma(M) \subset \mathbb{R}^2_+$. This excluded the point 0 from $co\sigma(M)$, hence establishes that M don't belong to $\mathcal{ST}(G)$.

Theorem 2.3. *Let q_p be the Jordan nilpotent operator of order p in a space of dimension p, and $q_p^{(j)}$ a direct sum of j elements q_p, then*

$$q_1 \oplus q_3^{(\infty)}$$

acting on

$$G = \mathcal{C} \oplus K,\ K = H \oplus H \oplus H,$$

(H separable) is similar to some operator, which is not finite.

Proof : This is to know that the operator $q_1 \oplus q_3^{(\infty)}$ comes from an operator $M(A)$ of Theorem 2.2. Now

$$M = M(A) = 0 \oplus T(A)$$

is similar to $M(0)$ by the similarity which comes from that used for $T(A)$, $0 \oplus S$. Choose in each factors of $\mathcal{G}$ an orthonormal basis, say

$$\{e_n, n \in N^*\}$$

in the first factor H,

$$\{f_n, n \in N^*\}$$

in the second factor H,

$$\{g_n, n \in N^*\}$$

in the third factor H,

Then $T(0)$ transforms e_n to 0, f_n to e_n and g_n to f_n. If

$$\varepsilon_n = Vect\ \{e_n, f_n, g_n\},$$

then ε_n reduces orthogonally $T(0)$, T_n the restriction of $T(0)$ to E_n is identically equal to q_3, q_1 is the operator 0 in $\mathbb{C}$, and we have

$$T\ (0) = q_1 \oplus q_4^{(\infty)}.$$

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ON QUASI EINSTEIN PSEUDO PROJECTIVE RICCI SYMMETRIC MANIFOLDS

BY

P.K. GHOSHAL AND SANDHYA GHOSHAL

1. INTRODUCTION

The notion of a Quasi Einstein manifold was introduced by Prof. M.C. Chaki and R.K. Maity [1] who defined it as follows :

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called Quasi Einstein if its Ricci tensor S of Type (0,2) is not identically zero and satisfies the condition

1) $S(X,Y) = ag(X,Y) + bA(X)A(Y)$

where a and b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

2) $g(X,U) = A(X)$ $\forall X$, U being a unit vector field.

In such a case, a and b are called associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

In 1988 Prof. M.C. Chaki introduced the notion of a Pseudo Ricci symmetric manifold [2] which was defined as follows :

A non-flat Riemannian manifold (M^n,g) $(n \geq 2)$ is called Pseudo Ricci symmetric if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition.

3) $(\nabla_X S)(Y,Z) = 2B(X)S(Y,Z) + B(Y)S(X,Z) + B(Z)S(X,Y)$ where B is a non-zero 1-form such that.

4) $g(X,\rho) = B(X)$ $\forall X$ and ∇ denotes the operator of covariant differentiation [5] with respect to the metric tensor of the manifold.

Subsequently, Prof. M.C. Chaki and S.K. Saha introduced a tensor called Projective Ricci tensor which they denoted by P and defined as follows :

5) $P(X,Y) = \frac{n}{n-1}\,[S(X,Y) - \frac{r}{n} g(X,Y)]$ [5]

where r denotes the scalar curvature.

In 1994 they defined a Pseude Projective Ricci symmetric manifold as follows : [4]

A non-flat Riemannian manifold (M^n,g) $(n>3)$ is said to be Pseudo Projective Ricci symmetric if its Projective Ricci tensor is not identically zero and satisfies the condition

6) $(\nabla_X P)(Y,Z) = 2D(X)\,P(Y,Z) + D(Y)\,P(Z,X) + D(Z)\,P(X,Y)$

where D is a non-zero 1-form such that

7) $g(X,V) = D(X)\ \forall X$

where V is a vector field not orthogonal to U and ∇ has the meaning already mentioned.

In 2001, Quasi Einstein Pseudo Ricci symmetric manifold was studied by Prof. M.C. Chaki and the first author of the present paper (P.K. Ghoshal) [3]

The object of this short paper is to extend the results of this paper by undertaking the study of Quasi Einstein Pseudo Projective Ricci symmetric manifolds. Such an n-dimensional manifold $(n>3)$ shall be denoted by the symbol $\Sigma(QE)_n$. For a manifold of this kind, the conditions (1), (2), (5), (6) and (7) will hold. In case of this type of manifold a,b will be called the associated scalars, A and D will be called the Primary and auxiliary associated 1-forms and U, V will be called the primary and auxiliary generators.

From 1) it follows that

8) $r = na + b$

In the present paper it is shown that in a $\Sigma(QE)_n$ the associated scalar b cannot be a non-zero constant. Further, the relation between the primary and auxiliary associated 1-forms and that between the primary and auxiliary generators are obtained in case of an admissible $\Sigma(QE)_n$.

1. Preliminaries

From (I) we get

(p. 1) $S(X,U) = (a+b)\,A(X)$

Again from (5) we have

$$\text{(p. 2)}\ P(X,U) = \frac{n}{n-1}[S(X,U) - \frac{(na+b)}{n} g(X,U)]$$

$$= bA(X)$$

Hence $P(X,U) \neq 0$

Again (p. 3) $P(U,U) = b$

From 6) it follows that

(p. 4) $(\nabla_X P)(Y,U) = 2D(X)\,P(Y,D) + D(Y)\,P(X,U) + D(U)\,P(X,Y)$

$$= 2b\,D(X)\,A(Y) + b\,D(Y)\,A(X) + D(U)\,P(X,Y)$$

From (p. 4) we get

(p.5)$(\nabla_X P)(U,U) = 2b[D(X) + D(U)A(X)]$

But

$(\nabla_X P)(U,U) = \nabla_X P(U,U) = \nabla_X b$ [by (p. 3)]

Hence (p. 5) can be expressed as follows

(p. 6) $\nabla_X b = 2b[D(X) + D(U)\,A(X)]$

These results will be used in the sequel

2. Non-Existence of A $\Sigma(QE)_n$, ($n \geq 3$) when b is a non zero constant

Since $b \neq 0$, it may be a non-zero constant or may not be a non-zero constant.

Let us suppose that b is a non-zero constant.

Then form (p. 6) it follows that

(2.1) $D(X) + D(U)\,A(X) = 0$

Putting $X=U$ in (2.1) we get

(2.2) $D(U) = 0$

Hence from (2.1) we have $D(X) = 0$ which is contray to be assumption that $D(X)$ is not-zero.

Hence a $\Sigma(QE)_n$ cannot exist if b is a non-zero constant. This leads to the following result.

Theorem 1 : In a $\Sigma(QE)_n$ ($n \geq 3$) the associated scalar b cannot be a non-zero constant.

3. $\Sigma(QE)_n$ in which the associated scalar b is not a non-zero constant

In this case we can express (p. 6) as follows :

(3.1)$X.\,b = 2b\,[D(X) + D(U)\,A(X)]$

or,

(3.2) $X.\log\sqrt{b} = D(X) + D(U)\,A(X)$

From (3.2) we have

(3.3) $D(X) = X.\log\sqrt{b} - D(U)\ A(X)$

From (3.3) it follows that

(3.4) $V = \text{grad}\ (\log\sqrt{b}) - D(U)\ U$

Considering (3.3) and (3.4) we can state that following result :

Theorem 2 : If in a $\Sigma(QE)_n$ the associated scalar b is not a non-zero constant, then the relation between its primary and auxiliary associated 1-forms A and D and that between its primary and auxiliary generators U and V are given by (3.3) and (3.4).

In conclusion, we offer our grateful thanks to Prof. M.C. Chaki for suggesting the problem and for his assistance in the preparation of this paper.

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"$\left|\overline{N}, p_n; \delta\right|_k$ SUMMABILITY OF A SERIES Σa_n"

NAMITA SHRIVASTAVA & SNIGDHA DUTTA

ABSTRACT : In the present paper a theorem of H.S. Özarslan [4] has been generalised under lighter conditions on sequences $\{p_n\}$ and $\{\lambda_n\}$ with $\{p_n\}$ as a sequence of positive real numbers instead of non-negative non-increasing sequence and a non-negative bounded sequence $\{\lambda_n\}$ instead of a non-negative, non-increasing sequence $\{\lambda_n\}$.

Key words : $\left|\overline{N}, p_n; \delta\right|_k$ summability, $[N, p_n; \delta]_k$ boundedness, summability factors, Hölder's inequality.

1. SUBJECT CLASSIFICATION NO.:

40F05 : 1980 AMS Mathematics (Absolute and Strong Summability).

2. NOTATION AND DEFINITIONS:

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with $\{S_n\}$ as the sequence of its partial sums. Let $\{P_n\}$ be a sequence of non - negative.

real numbers such that

$$P_n = \sum_{v-0}^{n} p_v \to \infty, \; as \; n \to \infty \qquad ..... (1)$$

$$(P_{-i} = p_{-i} = 0, \; i \geq 1)$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \qquad ..... (2)$$

defines the sequence $\{t_n\}$ of the $\left(\overline{N}, p_n\right)$ mean of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$

The series Σa_n is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$ (see [1]) if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\Delta t_{n-1}\right|^k < \infty \qquad \text{... (3)}$$

and it is said to be summable $\left|\overline{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$ if (see [3]3)

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\Delta t_{n-1}\right|^k < \infty \qquad \text{... (4)}$$

$$\text{where } \Delta t_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v,\ n \geq 1 \qquad \text{... (5)}$$

(For any sequence $\{r_n\}$, $\Delta r_n = r_n - r_{n+1}$)

In the special case when $\delta = 0$, $\left|\overline{N}, p_n; \delta\right|_k$ summability is the same as $\left|\overline{N}, p_n\right|_k$ summability.

The series Σa_n is said to be $\left[\overline{N}, p_n\right]_k$ bounded, if $k \geq 1$, (see [1])

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n) \text{ as } n \to \infty \qquad \text{... (6)}$$

and it is said to be $\left[\overline{N}, p_n; \delta\right]_k$ bounded, $k \geq 1$ if (see [2])

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k} p_v |s_v|^k = O(P_n) \text{ as } n \to \infty \qquad \text{... (7)}$$

If we take $\delta = 0$ then $\left[\overline{N}, p_n; \delta\right]_k$ boundeness is the same as $\left[\overline{N}, p_n\right]_k$ boundedness (see [1]).

INTRODUCTION

In this paper the following theorem A of H.S. Özarslan [4] is generalized under lighter conditions.

Theorem A [4]

Let $\{p_n\}$ be a positive non-increasing sequence such that $P_n \to \infty$ as $n \to \infty$ and

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{p_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right\} \qquad \ldots (8)$$

If Σa_n is $|\bar{N}, p_n; \delta|_k$ bounded and $\{\lambda_n\}$ is non-negative, non increasing sequence such that

$$\Sigma p_n \lambda_n < \infty \qquad \ldots (9)$$

and $P_n\ (\Delta\lambda_n) = O(p_n\lambda_n)$ ... (10)

then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$, $K \geq 1$

MAIN THEOREM

In this paper our aim to prove the following theorem [B] for sequences $\{p_n\}$ and $\{\lambda_n\}$ such that $\{p_n\}$ is of positive real numbers instead of non-negative, non-increasing sequence and a non-negative bounded sequence $\{\lambda_n\}$ instead of non-negative, non-increasing sequence $\{\lambda_n\}$ of theorem A.

THEOREM [B]

Let $\{p_n\}$ be a sequence of positive real numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty$, as $n \to \infty$, $(P_{-i} = p_{-i} = 0)$ $i \geq 1$ and $\{\lambda_n\}$ be a non-negative bounded sequence such that the conditions (7), (8) and (10) are satisfied and

$$\sum_{v=0}^{\infty}\frac{p_v}{P_v}\lambda_v^k < \infty \qquad \ldots (11)$$

and $\Sigma\left|\Delta\left(\lambda_v^k\right)\right| < \infty$ ... (12)

Then the series $\Sigma a_n \lambda_n$ is summable $\left|\overline{N}, p_n; \delta\right|_k$, $k \geq 1$

PROOF

Let $\{T_n\}$ denotes the $\left(\overline{N}, p_n\right)$ mean of the series $\Sigma a_n \lambda_n$. Then by definition (2) we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r$$

$$= \frac{1}{P_n} \sum_{v=0}^{n} \left(P_n - P_{v-1}\right) a_v \lambda_r$$

Then for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v \qquad (P_{-1} = 0) \quad \ldots (13)$$

By applying Able's transformation and simplifying we easily get the relation.

$$T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-1} P_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-1} s_v P_v \left(\Delta \lambda_v\right) + \frac{p_n s_n \lambda_n}{P_n} \quad \ldots (14)$$

$$= T_{n,1} + T_{n,2} + T_{n,3} \text{ say} \quad \ldots\ldots (15)$$

To prove the theorem, by using a well known inequality [6, Page 216]*, it is sufficient to show that (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|T_{n,r}\right|^k < \infty \qquad (\text{for } r = 1,2,3) \quad \ldots (16)$$

Applying Hölder's inequality with indices k and k' where

$\frac{1}{k} + \frac{1}{k'} = 1$ and using (7), (8), (11) and (12) we get

$$\sum_{n=1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|T_{n,1}\right|^k \leq \sum_{n=1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^{\,k}} \left\{\sum_{v=0}^{n-1} p_v \lambda_n |s_v|\right\}^k \quad \ldots (17)$$

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* Inequality in [6, Page 216]. If there are n positive quanities a,b,c.....k then $(a+b+c+\ldots.+k)^m < n^{k-1}\,(a^m + b^m + c^m + \ldots..+ k^m)$ for $m \geq 1$.

$$\leq \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\left[\left\{\frac{1}{P_{n-1}}\sum_{v=0}^{n-1} p_v \lambda_v^k |s_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=0}^{n-1} p_v\right\}^{k-1}\right]$$

$$\leq O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=0}^{n-1} p_v \lambda_v^k |s_v|^k$$

$$\leq O(1)\sum_{v=0}^{m} p_v(\lambda_v)^k |s_v|^k \sum_{r=v+1}^{m+1}\left(\frac{P_r}{p_r}\right)^{\delta k-1}\frac{1}{P_{r-1}}$$

$$\leq O(1)\sum_{v=0}^{m} p_v(\lambda_v)^k |s_v|^k \left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v} \qquad \ldots (18)$$

By virtue of (7)

$$= O(1)\sum_{v=0}^{m}\frac{(\lambda_v)^k}{P_v} p_v |s_v|^k \left(\frac{P_v}{p_v}\right)^{\delta k} \qquad \ldots (19)$$

By Abel's transformation

$$\sum_{v=0}^{m}\frac{(\lambda_v)^k}{P_v} p_v|s_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k} = \sum_{v=0}^{m-1}\left[\frac{(\lambda_v)^k}{P_v}-\frac{(\lambda_{v+1})^k}{P_{v+1}}\right]\sum_{r=0}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k} p_r|s_r|^k$$

$$+\frac{(\lambda_m)^k}{P_m}\sum_{r=0}^{m}(p_r)\left|s_r\right|^k\left(\frac{P_r}{p_r}\right)^{\delta k}$$

So that (17) reduces to

$$= O(1)\left[\Sigma\left\{\frac{(\lambda_v)^k}{P_v}-\frac{(\lambda_{v+1})^k}{P_v}+\frac{(\lambda_{v+1})^k}{P_v}-\frac{(\lambda_{v+1})^k}{P_{v+1}}\right\}P_v+\frac{(\lambda_m)^k}{P_m}P_m\right]$$

By virtue of (7).

$$= O(1)\left\{\sum_{v=0}^{m-1}\frac{1}{P_v}\left|\Delta\lambda_v\right|^k + \frac{(\lambda_{v+1})^k}{P_{v+1}}\frac{P_{v+1}}{P_v}\right\}P_v + O(1)(\lambda_m)^k$$

$$= O(1)\sum_{v=0}^{m-1}\left|\Delta\lambda_v\right|^k + \sum_{v=0}^{m}\frac{(\lambda_v)^k p_v}{P_v} + O\left(\lambda_m^k\right)$$

$= O(1)$, as $m \to \infty$

By virtue of (11), (12) and as $\{\lambda_n\}$ is bounded.

Again by using the condition (10) we have

$$\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|T_{n,2}\right|^k = \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=0}^{n-1}S_v(\Delta\lambda_v)P_v\right|^k$$

$$O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{(P_{n-1})^k}\left\{\sum_{v=0}^{n-1}p_v\lambda_v\left|S_v\right|\right\}^k$$

$= O(1)$, $m \to \infty$

As in case $T_{n,1}$ (see the proof from (17) onwards)

Finally as in $T_{n,1}$, we have

$$\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|T_{n,3}\right|^k = \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{p_n\lambda_n s_n}{P_n}\right|^k$$

$$= \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}(\lambda_n)^k\left|S_n\right|^k$$

$$= \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{P_n}{P_n}(\lambda_n)^k\left|S_n\right|^k$$

$$= \sum_{n=1}^{m+1} \left(\frac{P_n}{P_n}\right)^{\delta k} \frac{1}{P_n} \left[p_n (\lambda_n)^k |s_n|^k \right]$$

$$= O\ (1), \text{ as } m \to \infty$$

As in case $T_{n,1}$ (see the proof from (18) onward)

Therefore we have

$$\sum_{n=1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k = 0\ (1) \text{ as } m \to \infty \text{ for } r = 1,2,3$$

This completes the proof of the theorem.

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ON A SEMI-SYMMETRIC NON-METRIC CONNECTION IN A KENMOTSU MANIFOLD

B. PRASAD AND ASHWAMEDH MOURYA

ABSTRACT : Some properties of the curvature tensor. Weyl conformal curvature tensor, projective curvature tensor, concircular curvature tensor, W_2 curvature tensor and special curvature tensor of a semi-symmetric non-metric connection in a Kenniotsu manifold are studied.

1. INTRODUCTION

Let M be an n-dimensional almost contact metric manifold (Blair 1976) with an almost contact metric structure (ϕ,ξ,u,g), that is ϕ is a (1,1) tensor field, ξ is a vector field, u is a 1-form and g is a Riemannian metric on M such that

$$\phi^2 = -1 + u\otimes\xi, \quad u(\xi) = 1, \quad \phi\xi = 0, \quad u o\phi = 0 \qquad \text{.....} \ (1.1)$$

$$g(\phi X, \phi Y) = g(X,Y) - u(X)\, u(Y), \qquad \text{....} \ (1.2)$$

$$\Phi(X,Y) = g(\phi X,Y) = -\Phi\,(Y,X),\ g(\xi,X) = u(X), \qquad \text{....} \ (1.3)$$

for all X,Y, $\in$ TM. An almost contact metric manifold is called a Kenmotsu manifold (Kenmotsu, 1972) if

$$(\nabla_x\phi)Y = u(Y)\ \phi X - g\,(\phi X,Y)\xi, \qquad \text{....} \ (1.4)$$

where ∇ is Levi Civita connection of g from (1.4) it follows that

$$\nabla x\xi = \phi^2 X. \qquad \text{....} \ (1.5)$$

Also in Kenmotsu manifold, we have

$$(\nabla_x u)\,(Y) = g(X,Y) - u(X)\, u(Y)$$

A semi-symmetric non-metric connection $\overline{\nabla}$ in an almost contact metric manifold M can be defined by (Prasad and Verma 2004)

$$\overline{\nabla}_x Y = \nabla_x Y + u(Y)X + a(X)Y \qquad \text{.....} \ (1.6)$$

2. CURVATURE TENSOR

Let R and $\bar{R}$ be the curvature tensor of ∇ and $\bar{\nabla}$ in almost contact metric manifold. In view of (1.6), $\bar{R}$ and R are related by

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y + da(X,Y)Z, \quad \text{.... (2.1)}$$

where α is a (0.2) tensor defined by

$$\alpha(X,Y) - (\bar{\nabla}_X u)Y = -g(X,Y) - a(X)a(Y). \quad \text{.... (2.2)}$$

From (2.1), we deduce that

$$\bar{S}(X,Y) = S(X,Y) - (n-1)\alpha(X,Y) \quad \text{.... (2.3)}$$

$$\bar{r} = r - (n-1)\alpha, \quad \text{.... (2.4)}$$

where $\bar{S}$ (resp $\bar{r}$) and S(resp r) are Ricci tensor (scalar curvature) of $\bar{\nabla}$ and ∇ respectively.

Lemma (2.1) : Let M be a Kenmotsu manifold with the semi-symmetric non-metric connection $\bar{\nabla}$. Then

$$(\bar{\nabla}_X \phi)Y = \Phi(Y,X)\xi - a(X)\phi Y, \quad \text{.... (2.5)}$$

$$\bar{\nabla}_X \xi = u(X)\xi + a(X)\xi, \quad \text{.... (2.6)}$$

$$(\bar{\nabla}_X u)Y = -g(X,Y) - a(X)u(Y) = \alpha(X,Y), \quad \text{.... (2.7)}$$

$$\bar{R}(X,Y)Z = R(X,Y)Z+g(Y,Z)X-g(X,Z)Y+a(Y)u(Z)X-a(X)u(Z)Y+da(X,Y)Z \quad \text{.... (2.8)}$$

Proof : Using (1.6), (1.1), (1.4) and (1.3), we get (2.5). Equation (2.6) follows from (1.6) and (1.5) and (1.2) we get (2.7). In consequences of (2.1) and (2.7), we get (2.8).

Theorem (2.2) : In a Kenmotsu manifold with semi-symmetric non-metric connection $\bar{\nabla}$ we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y$$
$$= a(Y)u(Z)X + a(Z)u(X)Y + a(X)u(Y)Z$$
$$-a(X)u(Z)Y - a(Y)u(X)Z - a(Z)u(Y)X, \quad \text{.... (2.9)}$$

$$\bar{R}(X,Y,Z,W) + \bar{R}(Y,X,Z,W) = 0, \quad \text{.... (2.10)}$$

$$\bar{R}(X,Y,Z,W) - \bar{R}(Z,W,X,Y) = a(Y)\,u(Z)\,g(X,W) - a(X)\,u(Z)\,g(Y,W)$$
$$a(W)\,u(X)\,g(Z,Y) + a(Z)\,u(X)\,g(W,Y).$$

Proof : Using (2.8) and the first Bianchi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \qquad \ldots (2.11)$$

with respect to Levi Civita connection ∇, we get (2.9), From (2.8) we have

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)$$
$$+ a(Y)\,u(Z)\,g(X,W) - a(X)\,u(Z)\,g(Y,W) \qquad \ldots (2.12)$$

which implies (2.10) and (2.11)

Remarks : If in particular $a = 0$, then from (2.9), and (2.11) we get the expression for the curvature tensor $\bar{R}$ of the manifold with respect to $\bar{\nabla}$ and its properties as were obtained by Tripathi and Kakkar in their paper, 2001.

Theorem (2.3) : In an n-dimensional Kenmotsu manifold the Ricci tensor $\bar{S}$ and scalar curvature $\bar{r}$ of semi-symmetric non-meric connection $\bar{\nabla}$ are given by

$$\bar{S}(X,Y) = S(X,Y) + (n-1)g\,(X,Y) + (n-1)\,a(X)\,u(Y), \qquad \ldots (2.13)$$

and $\bar{r} = r + n(n-1) + (n-1)\,a(\xi)$ ... (2.14)

consequently $\bar{S}$ symmetric iff

$a(X)\,u(Y) = a(Y)\,u(X)$.

Proof : From (2.3) and (2.7) we get (2.13), (2.14) follows from (2.13). In view of (2.13), we have

$$\bar{S}(X,Y) - \bar{S}(Y,X) = (n-1)\,\{a(X)\,u(Y) - a(Y)\,u(X)\}, \qquad \ldots (2.15)$$

If $\bar{S}(X,Y)$ is symmetric the left hand side of (2.15) vanishes, hence we get

$$a(X)\,u(Y) = a(Y)\,u(X) \qquad \ldots (2.16)$$

which proves the theorem.

Theorem (2.4) : If a Kenmotsu manifold of dimension n admits semi-symmetric non-metric connection $\bar{\nabla}$ then a necessary and sufficient condition for the Ricci tensor $\bar{\nabla}$ to be skew symmetric is that the Ricci tensor of the Kenmotsu connection ∇ is given by (2.18).

Proof : Form (2.13) we get

$$\bar{S}(X,Y) + \bar{S}(Y,X) = 2S(X,Y) + 2(n-1)\ [g(X,Y) + a(X)\ u(Y)]. \qquad \text{... (2.17)}$$

If $\bar{S}(X,Y)$ is skew symmetric the left hand side of (2.17) vanishes and we get

$$S(X,Y) = -\ (n-1)\ [g(X,Y) + a(X)\ u(Y)]. \qquad \text{... (2.18)}$$

On the other hand if $S(X,Y)$ is given by (2.18) then from (2.17) we get

$$\bar{S}(X,Y) + \bar{S}(Y,X) = 0$$

which proves the theorem.

3. CONFROMAL CURVATURE TENSOR

Theorem : If a Kenmotsu manifold admits a semi-symmetric non-metric $\bar{\nabla}$, then a necessary and sufficient codnition for the conformal tensor C of the manifold (M^n) with respect to Riemannian connection and the conformal curvature tesor $\bar{C}$ of the manifold with respect to semi-metric connection to be equal is that 1-from a is zero.

Proof : Let $\bar{C}$ and C denote the conformal curvature tensor with respect to $\bar{\nabla}$ and ∇ respectively. Then we have

$$'\bar{C}(X,Y,Z,W) = '\bar{R}(X,Y,Z,W) - \frac{1}{n-2}\{\bar{S}(Y,Z)\ g(X,W) - \bar{S}(X,Z)\ g(Y,W)$$

$$+ \bar{S}(X,W)\ g(Y,Z) - \bar{S}(Y,W)\ g(X,Z)\} + \frac{\bar{r}}{(n-1)(n-2)}\{g(Y,Z)\ g(X,W)$$

$$-\ g(X,Z)\ g(Y,W)\}$$

and $$'C(X,Y,Z,W) = 'R(X,Y,Z,W) - \frac{1}{n-2}\ \{S(Y,Z)\ g(X,W) - S(X,Z)\ g(Y,W)$$

$$+ S(X,W)\ g(Y,Z) - S(Y,W)\ g(X,Z)\} + \frac{r}{(n-1)(n-2)}\ \{g(Y,Z)\ g(X,W)$$

$$-\ g(X,Z)\ g(Y,W)\} \qquad \text{... (3.2)}$$

where $'\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y,Z),W)$

and $'C(X,Y,Z,W) = g(C(X,Y,Z),W)$

In consequences (2.12), (2.13), (2.14), (3.1) and (3.2) we find

$$'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W) + \frac{a(\xi)}{n-2}\{g(Y,Z)\ g(X,W) - g(X,Z)\ g(Y,W)\}$$

$$- \frac{1}{n-2}\{a(Y)\ g(X,W) - a(X)\ g(Y,W)\}\ u(Z) - \frac{n-1}{n-2}\{a(X)\ g(Y,Z)$$

$$- a(Y)\ g(X,Z)\}\ u(W) \qquad \ldots (3.3)$$

It is clear from (3.3) if 1-form a = 0, then

$$'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W)$$

Conversely let us consider

$'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W)$ then from (3.3) we get

$$\frac{1}{n-2}[a(X)\ g(Y,W) - a(Y)\ g(X,W)]\ u(Z)$$

$$-\frac{n-1}{n-2}\ [a(X)\ g(Y,W) - a(Y)\ g(X,W)]\ u(W)$$

$$+\frac{a(\xi)}{n-2}\ [g(Y,Z)\ g(X,W) - g(X,Z)\ g(Y,W)] = 0$$

which yields $u(Z)\ a(Y) = 0$

Since 1-form $u \neq 0$. hence, 1-form $a = 0$,

which proves the theorem.

Remarks : If in particular a = 0 then from (3.3) we get $'\bar{C}(X,Y,Z,W) = 'C(X,Y,Z,W)$ i.e. the Weyl conformal curvature tensor of the manifold with respect to ∇ and semi-symmetric non-metric connection $\bar{\nabla}$ are equal. This is the corresponding theorem as were proved by Tripathi and Kakkar (2001).

From (3.3) we have the conformal curvature tensor with respect to semi-symmetric non-metric connection satisfying the following algebraic properties

$$\bar{C}(X,Y,Z) + \bar{C}(Y,X,Z) = 0 \qquad \ldots (3.4)$$

and $\bar{C}(X,Y,Z) + \bar{C}(Y,Z,X) + \bar{C}(Z,X,Y)$

$$= \frac{1}{n-2}[u(Z)\ \{a(X)Y - a(Y)X\} + u(X)\ \{a(Y)Z - a(Z)Y\}$$

$$+ u(Y)\ \{a(Z)X - a(X)Z\}] \qquad \ldots (3.5)$$

In particular

$\bar{C}(X,Y,Z) + \bar{C}(Y,Z,X) + \bar{C}(Z,X,Y) = 0$ if any only if $a(X)\ Y = a(Y)X$

4. PROJECTIVE CURVATURE TENSOR

Theorem (4.1) : If an Kenmotsu manifold admits a semi-symmetric non-metric connection $\bar{\nabla}$ then a necessary and sufficient condition for the projective curvature tensor P of the manifold (M^n) with respect to Riemannian connection and the projective curvature tensor $\bar{P}$ of the manifold with respect to semi-symmetric non-metric connection to be equal is that

$a(X)\ u(Y) = a(Y)\ u(X)$

Proof : If $\bar{P}$ and P denote the projective curvature tensor with respect to $\bar{\nabla}$ and ∇ respectively. Then (Mishra, 1984)

$$\bar{P}(X,Y,Z) = \bar{R}(X,Y,Z) + \frac{1}{n+1}[\bar{S}(X,Y) - \bar{S}(Y,X)]Z - \frac{n}{n^2-1}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]$$

$$- \frac{1}{n^2-1}[\bar{S}(Z,Y)X - \bar{S}(Z,X)Y] \qquad \ldots (4.1)$$

and $$P(X,Y,Z) = R(X,Y,Z) + \frac{1}{n+1}[S(X,Y) - S(Y,Z)]Z - \frac{n}{n^2-1}[S(Y,Z)X - S(X,Z)Y]$$

$$- \frac{1}{n^2-1}[S(Z,Y)X - S(Z,X)Y] \qquad \ldots (4.2)$$

In consequences of (2.8), (2.13), (4.1) and (4.2) we find

$$\bar{P}(X,Y,Z) = P(X,Y,Z) + \frac{1}{n+1}[\{a(Y)u(Z) - a(Z)u(Y)\}X + \{a(Z)u(X)$$

$$- a(X)\ u(Z)\}Y] + \frac{n-1}{n+1}\ [a(X)\ u(Y) - a(Y)\ u(X)]Z \qquad \ldots (4.3)$$

If $a(X)\ u(Y) = a(Y)\ u(X)$ then from (4.3) we get

$\bar{P}(X,Y,Z) = P(X,Y,Z)$

Converse is also true. Hence the theorem.

Remarks : If in particular $a = 0$ then from (4.3) we get $\bar{P}(X,Y,Z) = P(X,Y)Z$ i.e. the Weyl projective curvature tensor of the manifold with respect to ∇ and semi-symmetric non-metric connection $\bar{\nabla}$ are equal. This is the corresponding theorem as were proved by Tripathi and Kakkar in (2001).

Theorem (4.2) : If the Ricci tensor $\bar{S}$ of the semi-symmetric non-metric connection $\bar{\nabla}$ in a Kenmotsu manifold vanishes then the curvature tensor with respect to $\bar{\nabla}$ is equal to the projective curvature tensor of the mainfold

Iff $a(X)\ u(Y) = a(Y)\ u(X)$

Proof : In view of $\bar{S} = 0$ and (4.1) we have

$$\bar{P}(X,Y,Z) = \bar{R}(X,Y)Z \qquad \text{... (4.4)}$$

From (4.1) and (4.3) we get

$$\bar{R}(X,Y)Z = P(X,Y,Z) + \frac{1}{n+1}\ [\{a(Y)\ u(Z) - a(Z)\ u(Y)\}X + \{a(Z)\ u(X)$$

$$- a(X)\ u(Z)\}Y] + \frac{n-1}{n+1}\ [a(X)\ u(Y) - a(Y)\ u(X)]Z \qquad \text{... (4.5)}$$

If $a(X)\ u(Y) = a(Y)\ u(X)$ then from (4.4) we get

$R(X,Y)Z = P(X,Y,Z)$.

Converse can be proved from (4.5), hence the theorem.

From (4.3), we have the projective curvature tensor with respect to semi-symmetric non-metric connection satisfying by the following properties

$$\bar{P}(X,Y,Z) + \bar{P}(Y,X,Z) = 0$$

and $\bar{P}(X,Y,Z) + \bar{P}(Y,Z,X) + \bar{P}(Z,X,Y)$

$= \{a(Y)\ u(Z) - a(Z)\ u(Y)\}X + \{a(Z)\ u(X) - a(X)\ u(Z)\}Y$

$+ \{a(X)\ u(Y) - a(Y)\ u(X)\}Z$.

If in particular

$\bar{P}(X,Y,Z) + \bar{P}(Y,Z,X) + \bar{P}(Z,X,Y) = 0$ Iff $a(Y)\ u(Z) = a(Z)\ u(Y)$

5. CONCIRCULAR CURVATURE TENSOR

Theorem (5.2) : If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\bar{\nabla}$, then a necessary and sufficient condition for the concircular curvature tensor

V of the manifold (M^n) with respect to Riemannian connection and the concircular curvature tensor $\bar{V}$ of the manifold with respect to semi-symmetric non-metric connection to be equal that

$$a(\xi)\ g(Y,Z) = n\ a(Y)\ u(Z)$$

Proof : Let $\bar{V}$ and V denote the concircular curvature tensor with respect to $\bar{\nabla}$ and ∇ respectively. Then we have

$$\bar{V}(X,Y,Z) = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \qquad \ldots (5.1)$$

and $V(X,Y,Z) = R(X,Y)Z - \dfrac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],$ $\qquad \ldots (5.2)$

In consequences of (2.8), (2.14) and (5.2) and (5.3) we find.

$$\bar{V}(X,Y,Z) = V(X,Y,Z) - \frac{a(\xi)}{n}[g(Y,Z)X - g(X,Z)Y] + u(Z)\ [a(Y)X - a(X)Y] \qquad \ldots (5.3)$$

From (5.3) we get If $a(\xi)\ g(Y,Z) = n\ u(Z)\ a(Y)$

then $\bar{V}(X,Y,Z) = V(X,Y,Z)$ $\qquad \ldots (5.4)$

Conversely let us consider $\bar{V}(X,Y,Z) = V(X,Y,Z)$ then from (5.3), we get

$$u(Z)\ [a(Y)X - a(X)Y] - \frac{a(\xi)}{n}[g(Y,Z)X - g(X,Z)Y] = 0,$$

which gives $a(\xi)\ g(Y,Z) = n\ u(Z)\ a(Y)$.

Hence the theorem.

Remarks : If in particular $a = 0$ then from (5.3) we get $\bar{V}(X,Y,Z) = V(X,Y,Z)$ i.e. the concircular curvature tensor of rhe manifold with respect to ∇ and semi-symmetric non-metric connection $\bar{\nabla}$ are equal. This is the corresponding theorem as where proved by Tripathi and Kakkar in their paper 2001.

From (5.3), we have the concircular curvature tensor with respect to semi-symmetric non-metric connection satisfying by the following algebraic properties.

$\bar{V}(X,Y,Z) + \bar{V}(Y,X,Z) = 0$

$\bar{V}(X,Y,Z) + \bar{V}(Y,Z,X) + \bar{V}(Z,X,Y)$

$= u(X)\ [a(Z)Y] - a(Y)Z] + u(Y)\ [a(X)Z - a(Z)X] + u(Z)\ [a(Y)X - a(X)Y].$

If in particular

$\bar{V}(X,Y,Z) + \bar{V}(Y,Z,X) + \bar{V}(Z,X,Y) = 0$

Iff $a(Y)X = a(X)Y.$

6. W_2 CURVATURE TENSOR

Theorem (6.1) : If an Kenmotsu manifold admits a semi-symmetric non-metric connection $\bar{\nabla}$ then a necessary and sufficient condition for the W_2 curvature tensor (Mishra and Pokhriyal, 1970) of the manifold (M^n) with respect to Riemannian connection and the W_2 curvature tensor of the manifold with respect to semi-symmetric non-metric connection to be equal is that $g(Y,Z) = -a(Y)\ u(Z)$.

Proof :

$$'\bar{W}2\,(X,Y,Z,W) + {}'\bar{R}\,(X,Y,Z,W) + \frac{1}{n-1}[g(X,Z)S(Y,W) - g(Y,Z)S(X,W) \quad \ldots (6.1)$$

and $'W_2(X,Y,Z,W) + {}'R(X,Y,Z,W) + \frac{1}{n-1}[g(X,Z)S(Y,W) - g(Y,Z)S(X,W) \quad \ldots (6.2)$

In consequences of (2.12), (6.1) and (6.2) we find

$$'\bar{W}2\,(X,Y,Z,W) = {}'W_2(X,Y,Z,W) + g(X,W)\ [g(Y,Z) + a(Y)\ u(Z)]$$
$$- g(Y,W)\ [g(X,Z) + a(X)\ u(Z)] \quad \ldots (6.3)$$

If $'\bar{W}2\,(X,Y,Z,W) = W_2\ (X,Y,Z,W)$, then from (6.3), we get $g(Y,Z) = -a(Y)\ Y(Z)$ Conversely if $g(Y,Z) = -a(Y)\ u(Z)$ then from (6.3), we get

$'\bar{W}2\,(X,Y,Z,W) = W_2(X,Y,Z,W).$

Hence the theorem.

From (6.2) we have the $\bar{W}2$ curvature tensor with respect to semi-symmetric non-metric connection satisfying the following algebraic properties.

$$\bar{W}2\,(X,Y,Z) + \bar{W}2\,(Y,X,Z) = 0 \quad \ldots (6.4)$$

and $\overline{W}_2(X,Y,Z) + \overline{W}_2(Y,Z,X) + \overline{W}_2(Z,X,Y)$

$$= [a(Y)\,u(Z) - a(Z)u(Y)]X + [a(Z)\,u(X) - a(X)\,u(Z)]Y + a[(X)\,u(Y) - a(Y)\,u(X)]Z \quad \ldots (6.5)$$

In particular $\overline{W}_2(X,Y,Z) + \overline{W}_2(Y,Z,X) + \overline{W}_2(Z,X,Y) = 0$,

Iff $a(X)u(Y) = a(Y)\,u(X)$

7. SPECIAL CURVATURE TENSOR

Recently Singh and Khan in 1998 define a special curvature tensor of the type (1.3) by the relation

$$J(X,Y,Z) = R(X,Y,Z) + R(X,Z,Y) \quad \ldots (7.1)$$

Or equivalently

$$g(J(X,Y,Z),W) = g(R(X,Y,Z),W) + g(R(X,Z,Y),W) \quad \ldots (7.2)$$

$$'\overline{J}(X,Y,Z,W) = {}'\overline{R}(X,Y,Z,W) + {}'\overline{R}(X,Z,Y,W) \quad \ldots (7.3)$$

It is obvious that

$$J(X,Y,Z) = J(X,Z,Y)$$

and $J(X,Y,Z) + J(Y,Z,X) + J(Z,X,Y) = 0 \quad \ldots (7.4)$

Theorem : On a semi-symmetric non-metric connection in a Kenmotsu manifold satisfies the following special curvature tensor

(i) $\overline{J}(X,Y,Z) + \overline{J}(Y,Z,X) + \overline{J}(Z,X,Y) = 0$

(ii) $\overline{J}(X,Y,Z) + \overline{J}(Y,X,Z) = g(Y,Z)X + g(X,Z)Y - 2g(X,Y)Z + u(Y)\,[a(Z)X - a(X)Z] + u(X)\,[a(Z)Y - a(Y)Z]$,

Proof : Using (2.12) in (7.1) we get

$$'\overline{J}(X,Y,Z,W) = {}'J(X,Y,Z,W) + 2g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W) - g(X,Y)\,g(Z,W) + u(Z)\,[a(Y)\,g(X,W) - a(X)g\,(Y,W)] + u(Y)\,[a(Z)\,g(X,W) - a(X)\,g(Z,W)] \quad \ldots (7.5)$$

Using (7.4) and (7.5) we get

$$\overline{J}(X,Y,Z) + \overline{J}(Y,Z,X) + \overline{J}(Z,X,Y) = 0$$

where $\bar{J}(X,Y,Z) = \bar{R}(X,Y,Z) + \bar{R}(X,Z,Y)$

Similarly we have

$$\bar{J}(X,Y,Z) + \bar{J}(Y,X,Z) = g(Y,Z)X + g(X,Z)Y - 2g(X,Y)Z + u(Y)\,[a(Z)X - a(X)Z] + u(X)\,[a(Z)Y - a(Y)Z],$$

which proves the statement.

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COMMON FIXED POINT THEOREM FOR RATIONAL INEQUALITY IN A QUASI 2-METRIC SPACE

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Fisher [1], Sharma and Bajaj [2] obtained some fixed point theorems in a complete Quasi 2-metric Space.

In this paper we will prove a fixed point theorem for commuting pair of four self mappings satisfying rational inequalities in a Quasi 2-metric space.

1. INTRODUCTION

DEFINITION 1 :

A Quasi 2-metric space is a space X in which for each triple of points x, y, z there exists a real function $d\ (x,y,z)$ such that

$[M_1]$ to each pair of distinct points x, y, z

$$d\ (x, y, z) \neq 0$$

$[M_2]$ $d\ (x, y, z) = 0$ when atleast two of x, y, z are equal

$[M_3]$ $d\ (x, y, z) \leq d\ (x, y, u) + d\ (x, u, z) + d\ (u, y, z)$

for all x, y, z, u in X.

DEFINITION 2 :

A sequence $\{x_n\}$ in a Quasi 2-metric space (X, d) is said to be convergent at x if

$$\lim_{n \to \infty} d\ (x_n, x, z) = 0 \text{ for all z in X}$$

DEFINITION 3 :

A sequence $\{x_n\}$ in a Quasi 2-metric space (X, d) is said to be Cauchy sequence if

$$\lim_{m,\, n \to \infty} d\ (x_m, x_n, z) = 0 \text{ for all z in X}$$

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DEFINITION 4 :

A Quasi 2-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to some point of X.

THE MAIN RESULT

Let (X, d) be a Quasi 2-complete metric space and let E, F, T and S be four continuous self mappings of X satisfying

$$ES = SE,\ FT = TF,\ E(X) \subset T(X) \text{ and } F(X) \subseteq S(X) \qquad \text{... (1)}$$

$$\begin{aligned} d(Ex, Fy, a) \le\ & a_1 \frac{d(Sx, Ex, a)\left[d(Ty, Fy, a) + d(Ex, Ty, a)\right]}{d(Sx, Ty, a) + d(Ex, Ty, a)} \\ & + a_2 \frac{d(Ex, Ty, a)\left[d(Sx, Ex, a) + d(Ty, Fy, a)\right]}{\left[d(Sx, Ty, a) + d(Ex, Ty, a)\right]} \\ & + a_3 \frac{d(Ty, Fy, a)\left[1 + d(Sx, Ex, a)\right]}{1 + d(Sx, Ty, a)} \\ & + a_4 \frac{d(Sx, Ex, a)\, d(Ty, Fy, a)}{d(Sx, Ty, a)} \\ & + a_5 \left[d(Sx, Ex, a) + d(Ty, Fy, a)\right] \\ & + a_6\, d(Sx, Ty, a) \qquad \text{... (2)} \end{aligned}$$

for all $x, y \in X$ with $Sx \neq Ty$,

$d(Sx, Ty, a) + d(Ex, Ty, a) \neq 0$,

$a_1, a_2, a_3, a_4, a_5, a_6 \geq 0$,

$a_6 < 1$ and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$

Then E, F, T and S has a unique common fixed point.

PROOF

Let $x_0 \in X$, by (1) there exists a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Tx_{2n+1} = Ex_{2n}$ and

$y_{2n+1} = Sx_{2n+2} = Fx_{2n+1},\ n = 0, 1, 2, 3$

From (2), we have

$$
\begin{aligned}
& d(y_{2n}, y_{2n+1}, a) = d(Ex_{2n}, Fx_{2n+1}, a) \\
& \leq \frac{a_1 d(Sx_{2n}, Ex_{2n}, a)\left[d(Tx_{2n+1}, Fx_{2n+1}, a) + d(Ex_{2n}, Tx_{2n+1}, a)\right]}{d(Sx_{2n}, Tx_{2n+1}, a) + d(Ex_{2n}, Tx_{2n+1}, a)} \\
& + \frac{a_2 d(Ex_{2n}, Tx_{2n+1}, a)\left[d(Sx_{2n}, Ex_{2n}, a) + d(Tx_{2n+1}, Fx_{2n+1}, a)\right]}{d(Sx_{2n}, Tx_{2n+1}, a) + d(Ex_{2n}, Tx_{2n+1}, a)} \\
& + \frac{a_3 d(Tx_{2n+1}, Fx_{2n+1}, a)\left[1 + d(Sx_{2n}, Ex_{2n}, a)\right]}{1 + d(Sx_{2n}, Tx_{2n+1}, a)} \\
& + \frac{a_4 d(Sx_{2n}, Ex_{2n}, a)\, d(Tx_{2n+1}, Fx_{2n+1}, a)}{d(Sx_{2n}, Tx_{2n+1}, a)} \\
& + a_5\left[d(Sx_{2n}, Ex_{2n}, a) + d(Tx_{2n+1}, Fx_{2n+1}, a)\right] \\
& + a_6 d\,(Sx_{2n}, Tx_{2n+1}, a) \\
& \leq \frac{a_1 d(y_{2n-1}, y_{2n}, a)\left[d(y_{2n}, y_{2n+1}, a) + d(y_{2n}, y_{2n}, a)\right]}{d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n}, a)} \\
& + \frac{a_2 d(y_{2n}, y_{2n}, a)\left[d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a)\right]}{d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n}, a)} \\
& + \frac{a_3 d(y_{2n}, y_{2n+1}, a)\left[1 + d(y_{2n-1}, y_{2n}, a)\right]}{1 + d(y_{2n-1}, y_{2n}, a)} \\
& + \frac{a_4 d(y_{2n-1}, y_{2n}, a)\, d(y_{2n}, y_{2n+1}, a)}{d(y_{2n-1}, y_{2n}, a)} \\
& + a_5\left[d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a)\right] \\
& + a_6 d\,(y_{2n-1}, y_{2n}, a)
\end{aligned}
$$

$$
\begin{aligned}
& \leq (a_1 + a_3 + a_4 + a_5)\, d(y_{2n}, y_{2n+1}, a) \\
& \quad + (a_5 + a_6)\, d(y_{2n-1}, y_{2n}, a)
\end{aligned}
$$

Therefore,

$$d(y_{2n}, y_{2n+1}, a) \leq \frac{(a_5+a_6)}{[1-(a_1+a_3+a_4+a_5)]} d(y_{2n-1}, y_{2n}, a)$$

i.e. $d(y_{2n}, y_{2n+1}, a) \leq k\ d(y_{2n-1}, y_{2n}, a)$

where

$$k = \frac{(a_5+a_6)}{[1-(a_1+a_3+a_4+a_5)]} < 1$$

Now

$$\begin{aligned} d(y_n, y_{n+1}, a) &\leq k\ d(y_{n-1}, y_n, a) \\ &\leq k^n\ d(y_0, y_1, a) \end{aligned}$$

For every integer $p > 0$, we get

$$\begin{aligned} d(y_n, y_{n+p}, a) \leq{} & d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+p}, a) \\ & + d(y_n, y_{n+p}, a) \\ \leq{} & d(y_n, y_{n+1}, a) + d(y_{n+1}, y_{n+2}, a) + \ldots \\ & + d(y_{n+p-1}, y_{n+p}, a) \\ & + d(y_{n+1}, y_{n+2}, y_{n+p}) + d(y_{n+2}, y_{n+3}, y_{n+p}) + \ldots \\ & + d(y_{n+p-1}, y_{n+p}, y_{n+p}) \\ & \leq k^n d(y_0, y_1, a) + k^{n+1} d(y_0, y_1, a) + \ldots \\ & + k^{n+p-1} d(y_0, y_1, a) \\ & + k^n d(y_0, y_1, y_{n+p}) + \ldots + k^{n+p-2} d(y_0, y_1, y_{n+p}) \\ \leq{} & k^n(1+k+k^2+\ldots+k^{p-1})\ d(y_0, y_1, a) \\ & + k^n(1+k+k^2+\ldots+k^{p-2})\ d(y_0, y_1, y_{n+p}) \\ \leq{} & k^n\left(\frac{1}{1-k}\right)\left[d(y_0, y_1, a) + d(y_0, y_1, y_{n+p})\right] \end{aligned}$$

Letting $n \to \infty$, we have $d(y_n, y_{n+p}, a) \to 0$. Therefore $\{y_n\}$ is a Cauchy Sequence in X. Then as X is complete so $\{y_n\}_{n \in N}$ converges to some $u \in X$. So consequently the sub — sequence

$\{Ex_{2n}\}$, $\{Fx_{2n+1}\}$, $\{Tx_{2n}\}$ and $\{Sx_{2n+1}\}$

of $\{y_n\}$ also converges to the same point u. Now from (1) since E, F, T and S are continuous, such that

$$E(S(x_n)) \to Eu,$$

$$S(E(x_n)) \to Su,$$

$$F(T(x_n)) \to Fu, \text{ and } T(F(x_n)) \to Tu$$

$$Eu = Su, \; Fu = Tu \qquad \ldots (5)$$

from (1)

$$\begin{aligned} d(EEx_{2n}, Fx_{2n+1}, a) &\leq \frac{a_1 d(SEx_{2n}, EEx_{2n}, a)\left[d(Tx_{2n+1}, Fx_{2n+1}, a) + d(EEx_{2n}, Tx_{2n+1}, a)\right]}{d(SEx_{2n}, Tx_{2n+1}, a) + d(EEx_{2n}, Tx_{2n+1}, a)} \\ &+ \frac{a_2 d(EEx_{2n}, Tx_{2n+1}, a)\left[d(SEx_{2n}, EEx_{2n}, a) + d(Tx_{2n+1}, Fx_{2n+1}, a)\right]}{d(SEx_{2n}, Tx_{2n+1}, a) + d(EEx_{2n}, Tx_{2n+1}, a)} \\ &+ \frac{a_3 d(Tx_{2n+1}, Fx_{2n+1}, a)\left[1 + d(SEx_{2n}, EEx_{2n}, a)\right]}{1 + d(SEx_{2n}, Tx_{2n+1}, a)} \\ &+ \frac{a_4 d(SEx_{2n}, EEx_{2n}, a)\, d(Tx_{2n+1}, Fx_{2n+1}, a)}{d(SEx_{2n}, Tx_{2n+1}, a)} \\ &+ a_5\left[d(SEx_{2n}, EEx_{2n}, a) + d(Tx_{2n-1}, Fx_{2n-1}, a)\right] \\ &+ a_6 d\,(SEx_{2n}, Tx_{2n+1}, a) \end{aligned}$$

$$\begin{aligned} d(Eu, u, a) \leq a_1 &\frac{d(Su, Eu, a)\left[d(u, u, a) + d(Eu, u, a)\right]}{d(Su, u, a) + d(Eu, u, a)} \\ + a_2 &\frac{d(Eu, u, a)\left[d(Su, Eu, a) + d(u, u, a)\right]}{d(Su, u, a) + d(Eu, u, a)} \\ + a_3 &\frac{d(u, u, a)\left[1 + d(Su, Eu, a)\right]}{1 + d(Su, u, a)} \\ + a_4 &\frac{d(Su, Eu, a)\, d(u, u, a)}{d(Su, u, a)} \\ + a_5 &\left[d(Su, Eu, a) + d(u, u, a)\right] + a_6 d(Su, u, a) \end{aligned}$$

Therefore

$$d(Eu, u, a) < d(Eu, u, a) \text{ as } a_6 < 1$$

Which is a contradiction.

This gives $Eu = u = Su$ i.e. u is a fixed point of E and S.

Similarly we have $Fu = u = Tu$

So u is a common fixed point of E, F, S and T.

In order to establish the Uniqueness of the common fixed point, let v be another fixed point of E, F, T and S. Then

$$\begin{aligned} d(u, v, a) &= d(Eu, Fv, a) \\ &\leq a_1 \frac{d(Su, Eu, a)\left[d(Tv, Fv, a) + d(Eu, Tv, a)\right]}{d(Su, Tv, a) + d(Eu, Tv, a)} \\ &+ a_2 \frac{d(Su, Tv, a)\left[d(Su, Eu, a) + d(Tv, Fv, a)\right]}{d(Su, Tv, a) + d(Eu, Tv, a)} \\ &+ a_3 \frac{d(Tv, Fv, a)\left[1+d(Su, Eu, a)\right]}{1+d(Su, Tv, a)} \\ &+ a_4 \frac{d(Su, Eu, a)\, d(Tv, Fv, a)}{d(Su, Tv, a)} \\ &+ a_5 \left[d(Su, Eu, a) + d(Tv, Fv, a)\right] \\ &+ a_6 d\,(Su, Tv, a) \end{aligned}$$

$$d(u, v, a) \leq a_6 d\,(u, v, a)$$

which is a contradiction as $a_6 < 1$. Therefore $u = v$.

Hence u is the unique common fixed point of E, F, T and S.

This complete the proof.

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